ACCELERATION WAVE IN HYPERELASTIC MURNAGHAN MATERIAL

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Abstract
This paper derives an equation for the propagation velocity for the acceleration wave in the Murnaghan material. Based on the literature [4], we adopted constants \( l, m \) and \( n \) that describe the sandstone as a hyperelastic material. The numerical analysis of the uniaxial strain state showed very small differences in the velocity of propagation of the acceleration wave compared to the undeformed material.

Key words: acceleration wave, Murnaghan material, sandstone

INTRODUCTION
Considering the motion \( x' = x'(X'^a, t) \) we assume that the functions
\[
\frac{\partial x'(X'^a, t)}{\partial X'^a}; \quad \frac{\partial^2 x'(X'^a, t)}{\partial t}\frac{\partial X'^a}{\partial t}
\]
are continuous on the surface \( \Sigma_\alpha \) (Fig.1), while the second and higher derivatives are discontinuous, particularly the acceleration \( x'^{\alpha}_{\beta} \). The phenomena that occur on the surface \( \Sigma_\alpha \) are referred to as an acceleration wave or the weak discontinuity wave as opposed to the strong discontinuity wave, for which the first derivatives of the function \( x'(X'^a, t) \), i.e. \( x'^{\alpha}_{\sigma} \) and \( x'^{\alpha}_{\beta} \), are discontinuous. The jumps in the second derivatives of the function \( x'(X'^a, t) \) are given by:
\[
[[x'^{\alpha}_{\beta}\gamma]] = A^\alpha_{\tau} N_{\tau} N_{\beta}; \quad [[x'^{\alpha}_{\sigma}\gamma]] = -UA^\alpha_{\tau} N_{\tau}
\]
\[
[[x'^{\alpha}_{\gamma}\gamma]] = U^2 A^\alpha_{\tau}
\]
(2)

Because the terms \( x'^{\alpha}_{\beta\gamma} \) and \( N_{\tau} \) are tensors, the set of parameters \( A^\alpha_{\tau} \) is a vector. This vector defines the jumps in all the second derivatives of the function \( x'(X'^a, t) \) and is termed amplitude of the acceleration wave. Based on the analysis of the equations of motion
\[
A^\alpha_{\tau} x'^{\alpha}_{\beta\gamma} + q_{\gamma} + \rho_R b_{\gamma} = \rho_R \ddot{x}_{\gamma}
\]
(3)

where
\[
q_{\gamma} = \rho_R \frac{\partial^2 \sigma}{\partial x_{\alpha} \partial x_{\beta}}
\]
(4)
in the adopted Cartesian coordinate system \( \{X'^a\} \) and assuming that density \( \rho_R \) and the field of mass force \( b_{\gamma} \) are continuous, we obtain a condition for propagation of the acceleration wave in the reference configuration \( B_\alpha \):
\[
(Q_{\alpha \beta} - \rho_R U^2 \delta_{\alpha \beta}) A^\epsilon = 0
\]
(5)

\( Q_{\alpha \beta} \) is an acoustic tensor for the direction \( N \):
\[
Q_{\alpha \beta} = A^\alpha_{\epsilon} N_{\epsilon} N_{\beta}
\]
(6)

Due to the symmetry \( A^\alpha_{\epsilon} N_{\epsilon} = A^\alpha_{\epsilon} N_{\epsilon} \), the tensor \( Q_{\alpha \beta} \) is also symmetric. The equation (5) shows that the amplitude \( A^\alpha_{\epsilon} \) is an eigenvector, and the product \( \rho_R U^2 \) is an eigenvalue of the acoustic tensor \( Q_{\alpha \beta} \). Since \( Q_{\alpha \beta} = Q_{\beta \alpha} \), there are three mutually orthogonal amplitudes and three respective real squares of the propagation velocity. If the eigenvalues are positive, then there is a real \( U \) and the surface propagates. If the eigenvalues are negative, then \( A^\epsilon = 0 \) and the surface \( \Sigma_\alpha \) is not a discontinuity surface.

MURNAGHAN MATERIAL
The properties of a second-order nonlinear material, termed the Murnaghan material, were described in the paper [1]. The elastic potential for an isotropic material is given by:
\[
W(I_1, I_2, I_3) = \frac{l + 2m}{24} (I_1 - 3)^3 +
\frac{\lambda + 2\mu + 4m}{8} (I_1 - 3)^2 +
\frac{8\mu + n}{8} (I_1 - 3) \frac{m}{4} (I_1 - 3)(I_2 - 3) +
\frac{8\mu + n}{8} (I_2 - 3) \frac{n}{8} (I_1 - 1) - \frac{4\mu + n}{8} (I_2 - 3)^2
\]
(7)

where \( \lambda \) and \( \mu \) are Lamé’s constants, while \( l, m, n \) are the second-order elastic constants. According to Murnaghan, the constitutive equation (7) that describes elastic energy of an isotropic
compressible material might be used to calculate moderate strains. The examples of application of the Murnaghan material for the analysis of wave phenomena were discussed in the papers [2], [3]. The elastic constants for the sandstone (modeled in the present paper as a Murnaghan material) were adopted as in the study [4].

HOMOGENEOUS DEFORMATIONS FOR THE DIRECTION $X^1$

We consider a homogeneous static deformation of the compressible body. With the coinciding Cartesian coordinate systems $\{x^i\}$ and $\{X^\alpha\}$, this deformation can be given by the following equations:

$$
\begin{align*}
    x^1 &= \lambda_1 X^1; \\
    x^2 &= \lambda_2 X^2; \\
    x^3 &= \lambda_3 X^3
\end{align*}
$$

where $\lambda_1, \lambda_2, \lambda_3 = \text{const}$. The coordinate systems $\{x^i\}$ (spatial) and $\{X^\alpha\}$ (material) parameterize the same space. Hence the deformation gradient is independent of the coordinates $X^\alpha$ and the time $t$.

We assume that the elastic medium is subjected to uniaxial tension:

$$
\begin{align*}
    \epsilon_{11} &= \lambda_1; \\
    \epsilon_{12} &= \lambda_2; \\
    \epsilon_{13} &= \lambda_3 \quad (9)
\end{align*}
$$

For the adopted deformation, the deformation gradient and the deformation tensors are:

$$
\begin{align*}
    [X^\alpha] &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \\
    [B^\mu] &= \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}
\end{align*}
$$

The invariants of the deformation tensors are

$$
\begin{align*}
    I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
    I_2 &= \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \\
    I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2 
\end{align*} \quad (11)
$$

We assume that the discontinuity surface propagates in the direction which is perpendicular to the axis $X^3$. The unit vector normal to the surface $\Sigma_R$ (see Fig.1) at the material coordinates $\{X^\alpha\}$ has the components

$$
\mathbf{N} = [\cos \alpha, \sin \alpha, 0] \quad (12)
$$

The elastic energy $\sigma$ per unit mass in the reference configuration could be expressed as a function of the three invariants ($I_1, I_2, I_3$):

$$
\sigma = \sigma(I_1, I_2, I_3) \quad (13)
$$

In order to calculate the propagation velocity for the surface $\Sigma_R$, we use the condition of acceleration wave propagation for the reference configuration $B_R$. For the selected direction of propagation, the acoustic tensor $Q_{\alpha \beta}$ is given by:

$$
\begin{align*}
    Q_{\alpha \beta} &= A_{\alpha \beta}^{1/4} N_1 N_1 + A_{\alpha \beta}^{1/2} N_1 N_2 + \\
    &+ A_{\alpha \beta}^{3/4} N_2 N_1 + A_{\alpha \beta}^{3} N_2 N_2
\end{align*}
\quad (14)
$$

In order to calculate the above values, we need to use a first-order material tensor:

$$
\begin{align*}
    A_{\alpha \beta}^{\sigma \tau} &= \rho_R \cdot \sigma_{\alpha \beta}^{\sigma \tau} = \rho_R \cdot \frac{\partial \sigma}{\partial X^\alpha X^\beta}
\end{align*}
$$

Nonzero coordinates of the material tensor $A_{\alpha \beta}^{\sigma \tau}$ are

$$
\begin{align*}
    A_{11}^{11} &= 2 \rho_R \left[ \sigma_1 + 4 \sigma_2 \lambda_2^2 + \sigma_3 \lambda_3^2 \right] \\
    A_{12}^{12} &= \sigma_2 + 2 \rho_R \left[ \sigma_2 \lambda_2 \lambda_3 \right] \\
    A_{13}^{13} &= \sigma_3 + 2 \rho_R \left[ \sigma_3 \lambda_2 \lambda_3 \right] \\
    A_{22}^{12} &= A_{21}^{12} = 2 \rho_R \left[ \sigma_2 + \sigma_3 \lambda_3 \right] \\
    A_{23}^{13} &= A_{32}^{13} = -2 \rho_R \left[ \sigma_2 \lambda_2 + \sigma_3 \lambda_3 \right] \\
    A_{33}^{13} &= A_{31}^{13} = 2 \rho_R \left[ \sigma_3 + \sigma_2 \lambda_2 \right]
\end{align*} \quad (16)
$$
For further calculations we assume the elastic Murnaghan material, with elastic potential as present in the equation (7). For this material, the derivatives with respect to the invariants are given by:

\[
A^1 = 4\rho_k\lambda_1 A^2, \\
A^2 = 2\rho_k \left[ \sigma_1 + \sigma_2 + 4\sigma_{12}\lambda_2 \right] \left( \lambda_1^2 + \lambda_2^2 \right) + \left[ \sigma_1 A_1 + 2\sigma_{12} \lambda_2^2 \right]. \\
A^3 = 2\rho_k \left( \sigma_1 + \sigma_2 \lambda_2^2 \right). \\
A^4 = 2\rho_k \left( \sigma_1 + \sigma_2 \lambda_2^2 \right)
\]

The elastic Murnaghan material is characterized by five constants \((\lambda, \mu, l, m, n)\). The first two are the Lame's constants, and the other three are the second-order elastic constants. The derivatives of the assumed elastic potential with respect to invariants are, respectively [5]:

\[
\sigma_K = \frac{\partial \sigma}{\partial I_k}; \quad \sigma_{KL} = \frac{\partial^2 \sigma}{\partial I_k \partial I_L}
\]

\[
\sigma_{KLM} = \frac{\partial^3 \sigma}{\partial I_k \partial I_L \partial I_M}
\]

\(K, L, M = 1, 2, 3\)

With \(\rho_k U^2\) being an eigenvalue of the acoustic tensor \(Q_{ab}\), we derive propagation velocities \(U_1, U_2, U_3\) for the discontinuity surface from:

\[
\begin{vmatrix}
Q_{11} - \rho_k U^2 & Q_{12} & 0 \\
Q_{12} & Q_{13} - \rho_k U^2 & 0 \\
0 & 0 & Q_{33} - \rho_k U^2
\end{vmatrix} = 0
\]

We obtain, respectively:

\[
U_{1,2} = \frac{Q_{11} + Q_{12} + \sqrt{(Q_{11} - Q_{12})^2 + 4Q_{12}^2}}{2\rho_k}
\]

\[
U_3 = \frac{Q_{33}}{\rho_k}
\]

Finally:

\[
U_{1,2} = \frac{1}{2} \sqrt{\frac{2(\lambda + 3\mu) + \Theta \pm 2\sqrt{(\Theta \cos^2 \alpha \cdot \sin^2 \alpha + \Psi)}}{\rho_k}}
\]
\[ U_3 = \sqrt{\frac{\mu + \left( m + \lambda - \frac{n}{2} \sin^2 \alpha + 2\mu \cos^2 \alpha \right) \varepsilon_{11}}{\rho \varepsilon}} \]

where

\[ \Omega = \left[ 2(\lambda + \mu + (2l + m + \lambda + \mu)\varepsilon_{11}) \right]^2 \]

\[ \Theta = 2 \left( 2l + m + 2\lambda + 2\mu + (2(m + \lambda + \mu)\cos^2 \alpha + \varepsilon_{11} \right) \varepsilon_{11} \]

\[ \Psi = \left\lceil \left( \lambda + \mu \right) \cos 2\alpha + \left( m + 2\mu + 2(m + \mu + \lambda)\cos^2 \alpha + \varepsilon_{11} \right) \varepsilon_{11} \right\rceil^2 \]

NUMERICAL ANALYSIS

The constants \( \lambda, \mu, m, n, l \) for the sandstone adopted from [4] are

\[ \lambda = 1.237[GPa]; \quad \mu = 7.597[GPa] \]
\[ m = -99.400[GPa]; \quad n = -84.900[GPa] \]
\[ l = -97.800[GPa] \] (24)

The velocity of wave propagation is obtained from (22).

The diagram below illustrates the velocity of discontinuity surface \( \Sigma_R \) with strain \( \varepsilon_{11} = 10^{-3} \).

CONCLUSION

The numerical analysis presented in this paper demonstrated that the propagation velocities \( U_1, U_2, U_3 \) change periodically. Velocity diagrams (see Fig. 2) illustrate the range of \( \alpha \in \{0; \pi\} \). This leads to the conclusion that the distribution of velocities \( U_1, U_2 \) and \( U_3 \) is repeated with every \( \pi \). Furthermore, the numerical analysis showed that the velocity of the acceleration wave that propagates in the analyzed Murnaghan material depends on the initial deformation of the medium.

Comparison of the results obtained with the parameters typical of the undeformed material reveals some differences. However, these differences seem insignificant for the propagation of the acceleration wave in sandstone analyzed in the present study. The most noticeable differences can be observed for the velocity \( U_3 \) which is constant for \( \varepsilon_{11} = 0 \) and by about 6 m/s greater compared to \( \varepsilon_{11} = 10^{-3} \). Current measurement methodologies allow for recording differences in propagation velocities.

References