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MATHEMATICS 2

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MATHEMATICS 2

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Preface

Calculus is a branch of mathematics that introduces concepts and tools to describe and analyze functions. Although some parts of calculus were known to the ancient Greeks, Egyptians and Chinese, the modern version of calculus was largely developed in the 17^{th} century, independently by Issac Newton and Gottfried Leibniz. Calculus is not only an important branch of mathematics in its own right but also provides a rigorous mathematical foundation of physics, engineering and many other branches of science.

In our book Mathematics 1 we introduced differential calculus topics including limits, derivatives and indefinite integrals. In the present book we introduce the general concepts of integral calculus, look at the major applications and state and use the Fundamental Theorem of Calculus.

We begin this book with a short review of integration techniques for finding definite integrals. Following this we go through a series of geometric applications of definite integrals and further techniques for evaluating definite integrals and improper integrals. Then we begin the study of functions of more than one variable and extend the concepts of single variable calculus to functions of several variables. Ordinary differential equations are presented in Chapter 5. Finally, we focus on linear differential systems.

The dominant feature of this book is formalism. Definitions and theorems are stated precisely, and several results are proved at a high level of rigor. Each section begins with a theoretical introduction, includes definitions of the basic notions followed by propositions and a brief summary of rules or properties. Solved examples are used to explain the details of the calculations. Most sections end with several exercises. These will test students' understanding of the material that was covered in the section. The exercises are limited in number so that it is feasible to work through all of them. They have been carefully chosen so that a student who does most of them will be well prepared for applications of calculus in later courses.

We are indebted to the reviewers Mirka Miller and Joe Ryan both from the University of Newcastle, Australia, and Francesc A. Muntaner-Batle from Universitat Internacional de Cataluña, Barcelona, Spain. We would like to thank them for their comments and suggestions which led to significant revisions.

We are especially grateful to Marcela Lascsáková from Technical University in Košice for her unfailing support, keen eyes and attention to detail.

We welcome comments and suggestions from students using this book. In particular, we are interested in hearing about any typographical, mathematical, or formatting errors found in this book.

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Contents

CHAPTER 1

The Definite Integral

1.1 Definition of the definite integral

Let f be a function defined on an interval $\langle a, b\rangle$ and let $f(x) \geq 0$, for all $x \in \langle a, b\rangle$.

We will assume that f is bounded on $\langle a, b\rangle$, that is, we assume that there exist numbers m and M such that $m \leq f(x) \leq M$, for all $x \in \langle a, b\rangle$.

We call a set $D = \{x_0, x_1, \ldots, x_n\}$ a *partition* of the interval $\langle a, b\rangle$ if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Such a partition D divides $\langle a, b\rangle$ into n intervals, $\langle x_{i-1}, x_i\rangle$, of lengths

$$\Delta x_i = x_i - x_{i-1},$$

where $i = 1, 2, \ldots, n$.

For each such interval $\langle x_{i-1}, x_i\rangle$, let M_i be the smallest number such that $f(x) \leq M_i$, for all $x \in \langle x_{i-1}, x_i\rangle$ and let m_i be the largest number such that $f(x) \geq m_i$, for all $x \in \langle x_{i-1}, x_i\rangle$. Note that if f is continuous on $\langle a, b\rangle$ then M_i is the maximum value of f on $\langle x_{i-1}, x_i\rangle$ and m_i is the minimum value of f on $\langle x_{i-1}, x_i\rangle$.

Note that the rectangle with base $\langle x_{i-1}, x_i \rangle$ and height M_i is called a *circumscribed rectangle* and the rectangle with base $\langle x_{i-1}, x_i \rangle$ and height m_i is called an *inscribed rectangle*, see Figures 1.1 and 1.2.

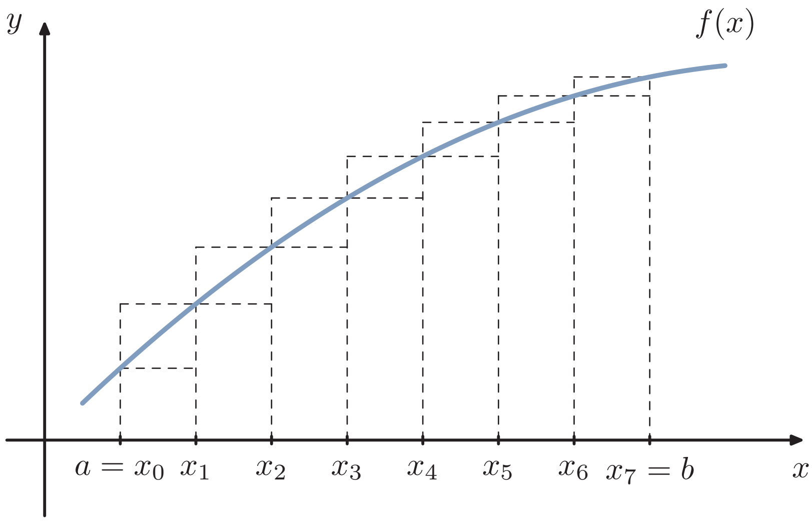

Figure 1.1: Circumscribed and inscribed rectangles for $n = 7$.

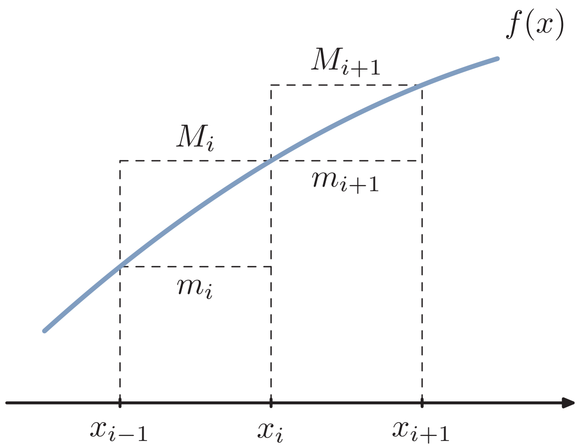

Figure 1.2: Maximum and minimum values on intervals.

Let

$$US_f(D) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = \sum_{i=1}^{n} M_i\Delta x_i \tag{1.1}$$

be the *upper sum* of f with respect to the partition D, and

$$LS_f(D) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = \sum_{i=1}^{n} m_i\Delta x_i \tag{1.2}$$

be the *lower sum* of f with respect to the partition D. Note that we always have

$$LS_f(D) \leq US_f(D). \tag{1.3}$$

According to Figure 1.1, we can see that $US_f(D)$ is the sum of the areas of the circumscribed rectangles for the partition D and $LS_f(D)$ is the sum of the areas of the inscribed rectangles for the partition D.

If we choose values $c_1, c_2, \ldots, c_n$ so that c_i is in the ith interval of the partition (that is, $x_{i-1} \leq c_i \leq x_i$) then

$$m_i \leq f(c_i) \leq M_i, \tag{1.4}$$

for $i = 1, 2, \ldots, n$, and so

$$LS_f(D) = \sum_{i=1}^{n} m_i\Delta x_i \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq \sum_{i=1}^{n} M_i\Delta x_i = US_f(D). \tag{1.5}$$

Definition 1.1.1. *(Integrable function, Definite integral)*
A function f is integrable on an interval $\langle a, b\rangle$ if there exists a unique number I such that

$$LS_f(D) \leq I \leq US_f(D), \tag{1.6}$$

for all partitions D of $\langle a, b\rangle$. If f is integrable on $\langle a, b\rangle$, we call I the definite integral of f on $\langle a, b\rangle$, which we denote

$$I = \int_a^b f(x)\, \mathrm{d}x. \tag{1.7}$$

The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \tag{1.8}$$

is called a *Riemann sum*. If we consider

$$\Delta x = \frac{b-a}{n}$$

as the length of the intervals $\langle x_{i-1}, x_i \rangle$, $i = 1, 2, \ldots, n$, then

$$\lim_{n\to\infty} \Delta x = 0.$$

For an integrable function f with points $c_1, c_2, \ldots, c_n$, where $x_{i-1} \le c_i \le x_i$ for all intervals, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\, \mathrm{d}x. \tag{1.9}$$

The basic notation for the definite integral is

$$I = \int_a^b f(x)\, \mathrm{d}x.$$

- The numbers a and b are called the *limits of integration*. The number a is the *lower limit* of integration and b is the *upper limit* of integration.
- The limits of integration form an interval $\langle a, b \rangle$. This interval is referred to as the *interval of integration*.
- The function $f(x)$ is called the *integrand*.
- The symbol $\mathrm{d}x$, the differential of x, plays the same role here as it did for the indefinite integral. The symbol $\mathrm{d}x$ tells us that x is the *variable of integration*.

There is a correspondence between the definite integral and the area of a region in the plane.

Definition 1.1.2. *(Area of region)*
Given an integrable function f with $f(x) \geq 0$, for all x in an interval $\langle a, b\rangle$, let R be the region in the plane bounded above by the curve $y = f(x)$, below by the interval $\langle a, b\rangle$ on the x-axis, and on the sides by the vertical lines $x = a$ and $x = b$. Then we define the area A of region R to be

$$A = \int_a^b f(x)\, \mathrm{d}x. \tag{1.10}$$

Figure 1.3 depicts the region beneath the graph of a function f over the interval $\langle a, b\rangle$.

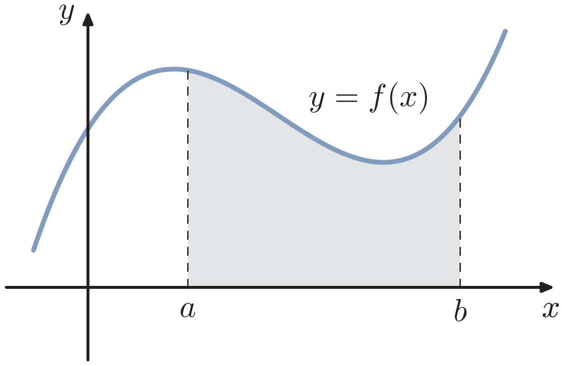

Figure 1.3: Region beneath f over $\langle a, b\rangle$.

Because of the complicated nature of the definition of the definite integral, one may wonder whether it is possible for any function to be integrable. We have the following theorem.

Theorem 1.1.1.
Any continuous function on an interval $\langle a, b\rangle$ is integrable on $\langle a, b\rangle$.

Figure 1.4 shows a function f defined as follows:

$$f(x) = \begin{cases} x^2, & \text{if } x < 1, \\ 3 - x, & \text{if } x \geq 1. \end{cases}$$

Since f is not continuous on $\langle 0, 2\rangle$, its integrability does not follow from Theorem 1.1.1. However, the function f is integrable on $\langle 0, 2\rangle$ because it is a piecewise continuous function.

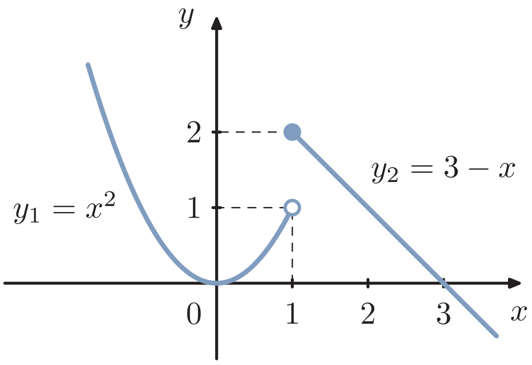

Figure 1.4: Example of the piecewise continuous function.

Definition 1.1.3. *(Piecewise continuous function)*
A function f is said to be piecewise continuous on an interval $\langle a, b\rangle$ if there is a partition $D = \{x_0, x_1, \ldots, x_n\}$ of $\langle a, b\rangle$ such that f is continuous on each open interval (x_{i-1}, x_i), $i = 1, 2, \ldots, n$, has limits from both the right and the left at each partition point x_i, $i = 1, 2, \ldots, n-1$, and has a right-hand limit at a and a left-hand limit at b.

Theorem 1.1.2.
If f is piecewise continuous function on $\langle a, b\rangle$ then f is integrable on $\langle a, b\rangle$.

1.2 Properties of the definite integral

In this section we give a survey of the general properties of the definite integral. These properties are very useful in the calculation of the integral.

Theorem 1.2.1. *(Homogeneous property)*
Suppose the function f is integrable over the interval $\langle a, b\rangle$ and k is an arbitrary constant. Then kf is integrable over the interval $\langle a, b\rangle$ and

$$\int_a^b kf(x)\ \mathrm{d}x = k\int_a^b f(x)\ \mathrm{d}x. \tag{1.11}$$

Theorem 1.2.2. *(Additive of integrand)*
Suppose the functions f and g are integrable over the interval $\langle a, b\rangle$. Then $f+g$ is integrable over the interval $\langle a, b\rangle$ and

$$\int_a^b \Big(f(x)+g(x)\Big)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x. \tag{1.12}$$

Of course, a similar statement is true for the difference of two functions.

Suppose f is integrable on $\langle a, b\rangle$ and c is a point with $a < c < b$. It may be shown that f is integrable on both $\langle a, c\rangle$ and $\langle c, b\rangle$. Moreover, using partitions which include c, we may write a Riemann sum for f over $\langle a, b\rangle$ as the sum of two Riemann sums, the first over the interval $\langle a, c\rangle$ and the second over the interval $\langle c, b\rangle$. Thus the total area from a to b $\left(\int_a^b f(x)\ \mathrm{d}x\right)$ is the area from a to c $\left(\int_a^c f(x)\ \mathrm{d}x\right)$ combined with the area from c to b $\left(\int_c^b f(x)\ \mathrm{d}x\right)$, see Figure 1.5.

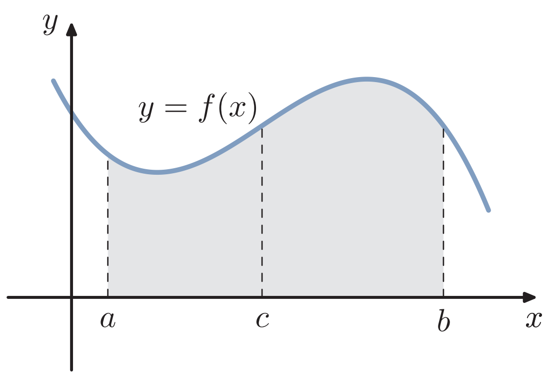

Figure 1.5: Illustration of the property (1.13).

Theorem 1.2.3. *(Additivity of limits)*
Suppose the function f is integrable over the interval $\langle a, b\rangle$ and let $c \in (a, b)$. Then

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x. \tag{1.13}$$

Theorem 1.2.4. *(Non-negativity of the integral)*
Suppose the function f is integrable over the interval $\langle a, b\rangle$ and that $f(x) \geq 0$ over $\langle a, b\rangle$. Then

$$\int_a^b f(x)\ \mathrm{d}x \geq 0. \tag{1.14}$$

Now, suppose f and g are both integrable on $\langle a, b\rangle$ and $g(x) \leq f(x)$ for all x in $\langle a, b\rangle$. It follows that for any given partition D, the upper sum of f will be greater than or equal to the corresponding upper sum of g. Since the definite integral is the largest number less than or equal to the value of any upper sum, the following theorem results.

Theorem 1.2.5. *(Monotone property of the integral)*
Suppose the functions f and g are integrable over the interval $\langle a, b\rangle$ and $g(x) \leq f(x)$, for all $x \in \langle a, b\rangle$. Then

$$\int_a^b g(x)\ \mathrm{d}x \leq \int_a^b f(x)\ \mathrm{d}x. \tag{1.15}$$

Theorem 1.2.6. *(Absolute integrability)*
Suppose the function f is integrable over the interval $\langle a, b\rangle$. Then $|f|$ is integrable over the interval $\langle a, b\rangle$ too, and

$$\left|\int_a^b f(x)\ \mathrm{d}x\right| \leq \int_a^b |f(x)|\ \mathrm{d}x. \tag{1.16}$$

Occasionally, we come across the problem of integrating over an interval of zero length. The following definition takes care of such a case.

Definition 1.2.1. *(Integral over interval of zero length)*
For any number belonging to the domain of f,

$$\int_a^a f(x)\ \mathrm{d}x = 0.$$

When we interchange the upper and lower limits of integration, we change the sign of the integral.

Definition 1.2.2. *(Reverse order of integration)*
Suppose the function f is integrable over the interval $\langle a, b\rangle$. Then

$$\int_a^b f(x)\ \mathrm{d}x = -\int_b^a f(x)\ \mathrm{d}x. \tag{1.17}$$

1.3 The fundamental theorem of calculus

The Fundamental Theorem of Calculus is a powerful weapon for evaluating definite integrals.

Theorem 1.3.1. *(Fundamental Theorem of Calculus)*
Let f be integrable over the interval $\langle a, b\rangle$. Let F be continuous on $\langle a, b\rangle$ and a primitive function of f on (a, b). Then

$$\int_a^b f(x)\ \mathrm{d}x = [F(x)]_a^b = F(b) - F(a). \tag{1.18}$$

Proof. Suppose f is integrable over the interval $\langle a, b\rangle$ and F is continuous there and F is a primitive function of f on (a, b). In particular,

$$F'(x) = f(x),$$

for all x in (a, b).

Let $D = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $\langle a, b\rangle$ and let

$$\Delta x_i = x_i - x_{i-1},$$

for $i = 1, 2, \ldots, n$. Now

$$
\begin{aligned}
F(b) - F(a) = F(x_n) - F(x_0) = F(x_n) + (F(x_{n-1}) - F(x_{n-1})) \\
+(F(x_{n-2}) - F(x_{n-2})) + \cdots + (F(x_1) - F(x_1)) - F(x_0) \\
= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots + (F(x_1) - F(x_0)) \\
= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})).
\end{aligned}
$$

By the Mean Value Theorem, for every $i = 1, 2, \ldots, n$, there exists a point c_i in the interval $\langle x_{i-1}, x_i \rangle$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}. \tag{1.19}$$

Since $F'(c_i) = f(c_i)$ and $x_i - x_{i-1} = \Delta x_i$, from (1.19) it follows that

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i. \tag{1.20}$$

Thus, using (1.20) we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{1.21}$$

Hence $F(b) - F(a)$ is equal to the value of a Riemann sum using the partition D, and so must lie between the upper and lower sums for D. That is, with respect to (1.5) we have that for any partition D

$$LS_f(D) \leq F(b) - F(a) \leq US_f(D).$$

But since f is integrable, there is only one number that has this property, namely, $I = \int_a^b f(x)\ \mathrm{d}x$, see (1.6). In other words, we have shown that

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

□

Example 1.3.1. Find the area under the parabola $y = x^2 + 1$ and above the interval $\langle -1, 2\rangle$ on the x-axis.

Solution The function $f(x) = x^2 + 1$ is continuous and positive on the interval $\langle -1, 2\rangle$. Since $F(x) = \frac{x^3}{3} + x$ is a primitive function of $f(x) = x^2 + 1$ then, with respect to (1.18), we have

$$\int_{-1}^{2} \left(x^2 + 1\right)\, \mathrm{d}x = \left[\frac{x^3}{3} + x\right]_{-1}^{2} = \left(\frac{2^3}{3} + 2\right) - \left(\frac{(-1)^3}{3} + (-1)\right) = 6.$$

Thus the area under the parabola $y = x^2 + 1$ and above the interval $\langle -1, 2\rangle$ on the x-axis is exactly 6 square units. See Figure 1.6. •

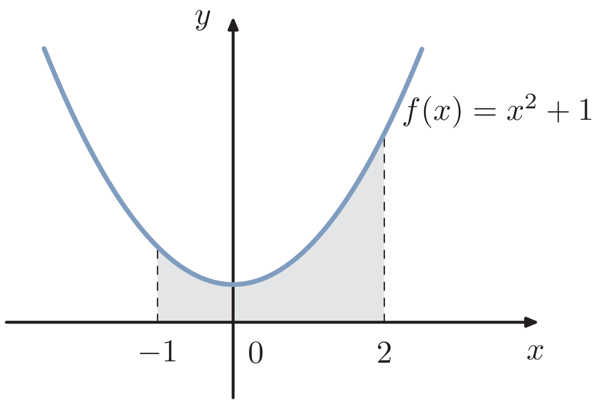

Figure 1.6: Region beneath the graph of $f(x) = x^2 + 1$ over the interval $\langle -1, 2\rangle$.

Example 1.3.2. Evaluate

$$\int_{e}^{e^2} \frac{1}{x \ln x}\, \mathrm{d}x.$$

Solution Since $F(x) = \ln|\ln x|$ is a primitive function of $f(x) = \frac{1}{x \ln x}$, we have

$$\int_{e}^{e^2} \frac{1}{x \ln x}\, \mathrm{d}x = [\ln|\ln x|]_{e}^{e^2} = \ln|\ln e^2| - \ln|\ln e| = \ln 2 - \ln 1 = \ln 2.$$

•

Example 1.3.3. Evaluate

$$\int_0^1 \frac{2x}{x^2+1}\,\mathrm{d}x.$$

Solution Since $F(x) = \ln(x^2+1)$ is a primitive function of $f(x) = \frac{2x}{x^2+1}$, we have

$$\int_0^1 \frac{2x}{x^2+1}\,\mathrm{d}x = \left[\ln(x^2+1)\right]_0^1 = \ln 2 - \ln 1 = \ln 2.$$

•

Example 1.3.4. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 x}\,\mathrm{d}x.$$

Solution We can see that the derivative of the denominator of the integrand gives the numerator with negative sign. This means

$$\left(1+\cos^2 x\right)' = -2\sin x\,\cos x.$$

Thus the primitive function of $f(x) = \frac{\sin 2x}{1+\cos^2 x}$ is $F(x) = -\ln\left(1+\cos^2 x\right)$.

Using (1.18) gives

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 x}\,\mathrm{d}x = -\left[\ln\left(1+\cos^2 x\right)\right]_0^{\frac{\pi}{2}}$$

$$= -\ln\left(1+\cos^2\frac{\pi}{2}\right) + \ln\left(1+\cos^2 0\right) = -\ln 1 + \ln 2 = \ln 2.$$

•

The value $\ln 2$ seems to be a general result for every definite integral. The next example shows that this is not true.

Example 1.3.5. Evaluate

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Solution The function $\arcsin x$ is the primitive function of the given integrand on $(-1,1)$ because $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.