TECHNICAL UNIVERSITY OF KOŠICE

Faculty of Mechanical Engineering

MATHEMATICS 1

Martin Bača

and

Andrea Feňovčíková

Edícia vedeckej a odbornej literatúry

Košice 2010

Technical University of Košice, Faculty of Mechanical Engineering

Martin Bača, Andrea Feňovčíková
MATHEMATICS 1

Editors:

Mirka Miller
School of Electrical Engineering and Computer Science
The University of Newcastle
NSW 2308, Australia

Francesc Antoni Muntaner-Batle
Facultat de Ciències Polítiques i Jurídiques
Universitat Internacional de Cataluña
Barcelona, Spain

Joe Ryan
School of Electrical Engineering and Computer Science
The University of Newcastle
NSW 2308, Australia

Press: C-PRESS, Košice

Printed in Slovakia

ISBN: 978-80-553-0472-4

Preface

The purpose of this book is to develop the student's understanding of the basic concepts of calculus. The book covers the first semester course of mathematics for foreign students at Technical University of Košice.

The text presents the fundamentals of calculus intuitively and clearly without sacrificing accuracy. The balance of written explanations and algebraic details found in the solutions to examples is the result of years of classroom experience. Students will find the text clear, concise, accurate, and readable.

Wherever possible, we have used figures to motivate new concepts. These figures are clearly drawn and carefully captioned and labeled to enhance student's comprehension.

Nearly 420 exercises are included to give students ample opportunity to test their understanding and apply their skills to real-world problems. Our experience shows that the student's understanding of calculus is linked directly to the number of problem-types solved successfully. We believe that our exercises will foster this success.

We would like to thank the reviewers Mirka Miller and Joe Ryan both from the University of Newcastle, Australia, and Francesc A. Muntaner-Batle from Universitat Internacional de Cataluña, Barcelona, Spain, who have read the manuscript. Their encouragement, comments, and suggestions have been most helpful.

Every effort has been made to verify that this text is free from error. The examples and exercises have been solved and checked independently by several people. We are grateful for their keen eyes and attention to detail. Our special thanks goes to Marcela Lascsáková from Technical University in Košice. However, the authors alone should be blamed for any remaining errors and imperfections.

Martin Bača
Andrea Feňovčíková

Contents

CHAPTER 1

Elementary Linear Algebra

1.1 Matrices

A rectangular array of numbers is called a *matrix* (plural, *matrices*). Matrices will usually be denoted by capital letters and the equation $A = [a_{ij}]$ means that the element in the i-th row and j-th column of the matrix A equals a_{ij}. We ordinarily write brackets around matrices, although this is not necessary when calculating with matrices. The *rows* of a matrix are horizontal, and the *columns* are vertical.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix of m rows and n columns is called a matrix with *dimensions* $m \times n$, or of *size* $m \times n$. The following are matrices with dimensions 3×2, 3×4 and 2×1:

$$\begin{bmatrix} 5 & -2 \\ 2 & 1 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 6 & 2 \\ 3 & 2 & -4 & 3 \\ 1 & -2 & 0 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

A matrix with dimensions $n \times n$ is called a *square matrix*. The elements $a_{11}, a_{22}, \ldots, a_{nn}$ of a square matrix are called the *diagonal elements.*

$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & -2 & 1 \\ 10 & -3 & 0 \end{bmatrix}$ is a square matrix as it has 3 rows and 3 columns. The diagonal elements of A are $a_{11} = 3$, $a_{22} = -2$ and $a_{33} = 0$.

Definition 1.1.1. *(Equality of matrices)*
Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if A and B have the same dimensions $m \times n$ and corresponding elements are equal, that is,

$$a_{ij} = b_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 1.1.1. What would make $A = \begin{bmatrix} 2 & 7 \\ -3 & 5 \end{bmatrix}$ to be equal to matrix $B = \begin{bmatrix} 2 & b_{12} \\ -3 & b_{22} \end{bmatrix}$?

Solution The two matrices A and B would be equal if $b_{12} = 7$ and $b_{22} = 5$.

•

Definition 1.1.2. *(Transpose of matrices)*
Let A be a matrix with dimensions $m \times n$. Matrix A^T is called the transpose of the matrix A if

$$a_{ij}^T = a_{ji}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

The transpose A^T of a matrix A is the matrix obtained from A by writing its rows as columns. Note that if A is a matrix with the dimensions $m \times n$ then A^T is a matrix with the dimensions $n \times m$.

Example 1.1.2. Find the transpose of the matrices.

a) $\begin{bmatrix} 4 \\ -3 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \end{bmatrix}.$

b) $\begin{bmatrix} 2 & 4 & -3 \\ 5 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 2 & 3 \\ -3 & 0 & 1 \end{bmatrix}.$

•

Definition 1.1.3. *(Addition of matrices)*
Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices with the same dimensions $m \times n$. Then $C = A + B$ is the matrix obtained by adding corresponding elements of A and B, that is,

$$c_{ij} = a_{ij} + b_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 1.1.3. $\begin{bmatrix} 5 & -2 \\ 2 & 1 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & 5 \\ 0 & 2 \end{bmatrix}.$

•

A matrix having zeros for all of its members is called a *zero matrix* and is often denoted by O. When a zero matrix is added to another matrix of the same dimensions, that same matrix is obtained. Thus a zero matrix is an *additive identity*. The additive inverse of a matrix can be obtained by replacing each member by its additive inverse. When two matrices that are inverses of each other are added, a zero matrix is obtained.

Example 1.1.4. $\begin{bmatrix} 5 & -2 \\ 2 & 1 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ -2 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

•

If we denote matrices by A and B and the *additive inverse* of B by $-B$ then we can state

$$A - B = A + (-B).$$

Definition 1.1.4. *(Scalar multiple)*
The product of a number k and a matrix $A = [a_{ij}]$ is the matrix, denoted $C = kA$, obtained by multiplying each number in A by the number k, that is,

$$c_{ij} = ka_{ij}$$

for $1 \le i \le m$ and $1 \le j \le n$.

Example 1.1.5. Consider the matrix $A = \begin{bmatrix} 3 & 12 & 0 \\ -9 & 6 & -3 \\ 0 & 9 & 15 \end{bmatrix}$.

Then

$$(-2)A = \begin{bmatrix} -6 & -24 & 0 \\ 18 & -12 & 6 \\ 0 & -18 & -30 \end{bmatrix} \text{ and } \frac{1}{3}A = \begin{bmatrix} 1 & 4 & 0 \\ -3 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix}.$$

•

Let A, B and C be matrices with the same dimensions $m \times n$, and k, t be arbitrary numbers. The matrix operations satisfy the following laws of arithmetic:

Additive commutative law

$$A + B = B + A \tag{1.1}$$

Additive associative law

$$(A + B) + C = A + (B + C) \tag{1.2}$$

Additive identity law

$$A + O = O + A = A \tag{1.3}$$

Additive inverse law

$$A + (-A) = (-A) + A = O \tag{1.4}$$

Distributive laws

$$(k+t)A = kA + tA \quad \text{and} \quad (k-t)A = kA - tA \tag{1.5}$$

$$k(A+B) = kA + kB \quad \text{and} \quad k(A-B) = kA - kB \tag{1.6}$$

$$k(tA) = (kt)A \tag{1.7}$$

Scalar unit

$$1A = A \quad \text{and} \quad (-1)A = -A \tag{1.8}$$

Scalar zero

$$0A = O \tag{1.9}$$

$$kA = O \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad A = O \tag{1.10}$$

Definition 1.1.5. *(Matrix product)*
Let $A = [a_{ij}]$ be a matrix of dimensions $m \times n$ and $B = [b_{jk}]$ be a matrix of dimensions $n \times r$. Then AB is the matrix $C = [c_{ik}]$ of dimensions $m \times r$ whose element c_{ik} is defined by the formula

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

for $1 \le i \le m$ and $1 \le k \le r$.

Example 1.1.6. Find the product of the matrices.

a) $$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 3\cdot 6 & 1\cdot 4 + 3\cdot 8 \\ 5\cdot 2 + 7\cdot 6 & 5\cdot 4 + 7\cdot 8 \end{bmatrix} = \begin{bmatrix} 20 & 28 \\ 52 & 76 \end{bmatrix}.$$

b) $$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 4\cdot 5 & 2\cdot 3 + 4\cdot 7 \\ 6\cdot 1 + 8\cdot 5 & 6\cdot 3 + 8\cdot 7 \end{bmatrix} = \begin{bmatrix} 22 & 34 \\ 46 & 74 \end{bmatrix}.$$

c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 9 & -1 & 2 & 0 \\ 0 & 3 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 5 & -8 & 14 \\ 27 & 9 & -14 & 28 \\ 45 & -5 & 10 & 0 \end{bmatrix}.$

•

Let A, B and C be matrices and k be an arbitrary number. Matrix multiplication obeys many of the familiar laws of arithmetic apart from the commutative law.

Multiplicative associative law: If A, B, C are matrices with the dimensions $m \times n$, $n \times r$, $r \times q$, respectively, then

$$(AB)C = A(BC). \tag{1.11}$$

If A, B are matrices with the dimensions $m \times n$, $n \times r$, respectively, then

$$k(AB) = (kA)B = A(kB), \tag{1.12}$$

$$A(-B) = (-A)B = -(AB). \tag{1.13}$$

Right distributive law: If A and B are matrices with the dimensions $m \times n$ and C is the matrix with the dimensions $n \times r$ then

$$(A + B)C = AC + BC. \tag{1.14}$$

Left distributive law: If D is the matrix with the dimensions $q \times m$ and A, B are matrices with the dimensions $m \times n$ then

$$D(A + B) = DA + DB. \tag{1.15}$$

Multiplicative identity law

$$AI = IA = A \tag{1.16}$$

Multiplication by zero matrix

$$OA = AO = O \tag{1.17}$$

In general, AB does not equal BA – matrix multiplication is not commutative!

Let A be a matrix with dimensions $m \times n$ and B be a matrix with dimensions

$r \times s$. If the product AB exists then the number of columns of matrix A has to be the same as the number of rows of matrix B; thus $n = r$ and the size of AB is $m \times s$. If the product BA exists then the number of columns of matrix B has to be the same as the number of rows of matrix A; this means $s = m$ and the size of BA is $r \times n$. For $AB = BA$, the resulting matrix has to be of the same size. This is only possible if A and B are square and are of the same size, $m = n = r = s$.

Even then in general

$$AB \neq BA.$$

Exercises

1.1. Find the dimensions of each matrix.

a) $A = \begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 0 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -1 & 2 & -3 & 4 \\ 2 & -4 & 6 & 8 \\ -2 & 4 & -6 & 8 \end{bmatrix}$.

c) $C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}$.

d) $D = \begin{bmatrix} 0 & -2 & 1 & 3 & 2 & 0 \end{bmatrix}$.

1.2. Let $A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -3 & 6 \end{bmatrix}$.

a) Find $A + B$. **b)** Find $A - B$. **c)** Find $B + A$.

d) Find $B - A$. **e)** Find $2A + 3B$. **f)** Find $5B - 4A$.

g) Show that $(A + B)(A - B) \neq A^2 - B^2$.

h) Show that $(A + B)(A + B) \neq A^2 + 2AB + B^2$.

i) Show that $(A+B)(A-B) = A^2 + BA - AB - B^2$.

j) Show that $(A+B)(A+B) = A^2 + BA + AB + B^2$.

1.3. Add. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 1 \\ 5 & 7 & 1 & 3 \\ 7 & 1 & 3 & 5 \end{bmatrix}$.

1.4. Subtract. $\begin{bmatrix} 6 & 4 & 2 \\ 4 & 2 & 6 \\ 2 & 6 & 4 \\ 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \\ 6 & 4 & 2 \end{bmatrix}$.

1.5. Compute these products.

a) $k \begin{bmatrix} 2 & -3 & 2k \\ -4 & 0 & 7 \\ \frac{1}{k} & 1 & 0 \\ -1 & k^2 & 5 \end{bmatrix}$. **b)** $xy \begin{bmatrix} \frac{x}{y} & \frac{y}{x} & -1 \\ \frac{-x^2}{y} & \frac{y}{x^2} & 1 \\ \frac{x}{y^2} & \frac{y^2}{x} & 0 \end{bmatrix}$.

1.6. Multiply. $\begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 & -3 & 0 & -2 \\ 4 & 2 & 7 & 1 \end{bmatrix}$.

1.7. Multiply. $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$.

1.8. Find AB and BA if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1.9. Multiply.

a) $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{bmatrix}$. **b)** $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ -2 & 2 \end{bmatrix}$.

1.10. Find AB and BA and compare.

$$A = \begin{bmatrix} 3 & 2 & -3 \\ -3 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.11. Compute.

a) $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}^2$. **b)** $\begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}^3$. **c)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^4$.

1.12. Compute.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

1.13. For a given matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, find every matrix B for which $AB = BA$.

1.2 Determinants

With every square matrix there is associated a number called the *determinant*. For a matrix A, the determinant is denoted by $|A|$ or $\det(A)$. For the trivial case of a one-by-one matrix the determinant is just the number in the matrix.

Definition 1.2.1. *(The determinant 2×2)*
The determinant of the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *is denoted* $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ *and is defined as follows:*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 1.2.1. Evaluate.

$$\begin{vmatrix} 5 & -2 \\ 2 & 1 \end{vmatrix} = 5 \cdot 1 - (-2) \cdot 2 = 9.$$

•

Example 1.2.2. Evaluate.

$$\begin{vmatrix} a-b & a+b \\ a+b & a-b \end{vmatrix} = (a-b)^2-(a+b)^2 = a^2-2ab+b^2-a^2-2ab-b^2 = -4ab.$$

•

Example 1.2.3. Evaluate.

$$\begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x = \cos 2x.$$

•

Example 1.2.4. Evaluate.

$$\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix} = \sin^2 x - (-\cos^2 x) = \sin^2 x + \cos^2 x = 1.$$

•

Definition 1.2.2. *(The determinant 3×3)*
The determinant of a three-by-three matrix is defined as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Example 1.2.5. Evaluate.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 1\cdot 5 - 2\cdot 1 + 3\cdot(-7) = -18.$$

•

Definition 1.2.3. *(Minor)*
In a matrix $A = [a_{ij}]$, the minor M_{ij} of an element a_{ij} is the determinant of the matrix found by deleting the i-th row and j-th column.

Example 1.2.6. Find M_{11} and M_{32} in the following matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 5 & 4 \\ 7 & -1 & 0 \end{bmatrix}.$$

Solution To find M_{11}, we delete the first row and the first column and we calculate the determinant of the matrix formed by remaining elements:

$$M_{11} = \begin{vmatrix} 5 & 4 \\ -1 & 0 \end{vmatrix} = 5 \cdot 0 - 4 \cdot (-1) = 4.$$

To find M_{32}, we delete the third row and the second column and we calculate the determinant of the matrix formed by remaining elements:

$$M_{32} = \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot (-2) = 8.$$

•

Definition 1.2.4. *(Cofactor)*
In a matrix $A = [a_{ij}]$, the cofactor of an element a_{ij} is denoted A_{ij} and is given by

$$A_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the minor of a_{ij}.

Example 1.2.7. In the matrix A given in Example 1.2.6, find cofactor A_{11} and A_{32}.

Solution In Example 1.2.6, we found that $M_{11} = 4$ and $M_{32} = 8$. Therefore,

$A_{11} = (-1)^2 M_{11} = 4$ and $A_{32} = (-1)^5 M_{32} = -8$. •

Definition 1.2.5. *(The determinant $n \times n$ – Laplace expansion)*
The determinant of the matrix $A = [a_{ij}]$ with the dimensions $n \times n$ can be found by using the so-called j-column Laplace expansion:
$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij}.$
or i-row Laplace expansion:
$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}.$

For example, if $A = [a_{ij}]$ is a matrix with the dimensions 3×3, the 1-column Laplace expansion gives

$$\begin{aligned} |A| =& a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\ =& a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ =& a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{21}\,(a_{12}\,a_{33} - a_{13}\,a_{32}) \\ & + a_{31}\,(a_{12}\,a_{23} - a_{13}\,a_{22}) \\ =& a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{21}\,a_{12}\,a_{33} \\ & + a_{21}\,a_{13}\,a_{32} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{13}\,a_{22}. \end{aligned}$$

Example 1.2.8. Evaluate the following determinant by using the 1-column Laplace expansion.

Solution

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix} = \sum_{i=1}^{4}(-1)^{i+1}a_{i1}M_{i1} = (-1)^2\;1\begin{vmatrix} 2 & 3 & 4 \\ 4 & 9 & 16 \\ 8 & 27 & 64 \end{vmatrix}$$

$$+(-1)^3\;1\begin{vmatrix} 1 & 1 & 1 \\ 4 & 9 & 16 \\ 8 & 27 & 64 \end{vmatrix} + (-1)^4\;1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 8 & 27 & 64 \end{vmatrix} + (-1)^5\;1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{vmatrix} = 12.$$

•