TECHNICAL UNIVERSITY OF KOŠICE

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MATHEMATICS 3

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MATHEMATICS 3

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Preface

The goal of this text is to help students learn to use calculus intelligently for solving a wide variety of mathematical and physical problems. This book Mathematics 3 covers the third semester course of mathematics for foreign students at Technical University of Košice.

In the first book Mathematics 1 we introduced differential calculus topics including limits, derivatives and indefinite integrals. In the second book Mathematics 2 we introduced the general concepts of integral calculus and gave techniques and applications of integration.

We begin the present book with double integrals including transformations and applications. Following this we go through triple integrals and through transformations in triple integrals. The next two chapters are devoted to line and surface integrals and there are presented Green's, Gauss's and Stokes' theorems. Infinite series are studied in Chapter 5. Finally, we focus on Fourier series.

This book contains numerous examples and illustrations to help make concepts clear. Solved examples are used to explain the details of the calculations. Most sections end with carefully chosen exercises which give students simple opportunity to test their understanding and apply their skills to real-world problems.

We would like to thank the reviewers Petr Kovář from Technical University of Ostrava, Czech Republic, and Yuqing Lin and Francesc A. Muntaner-Batle both from the University of Newcastle, Australia. We would like to thank them for their comments and suggestions which led to significant revisions.

The examples and exercises have been solved and checked independently by several people. Nevertheless we welcome comments and suggestions from students using this book. In particular, we are interested in hearing about any typographical, mathematical, or formatting errors found in this book.

Authors

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Contents

CHAPTER 1

Double Integrals

1.1 Definition of double integral

Let $f(x, y)$ be a function of two variables whose domain is a region R. A double integral is an integral of the function $f(x, y)$ over the region R. In this section, we define double integrals and show tools for their evaluation.

If for every point $[x, y] \in R$ is $f(x, y) > 0$, then the double integral is equal to the volume of the solid under the surface $z = f(x, y)$ and above the xy-plane restricted to in the region of integration R, see Figure 1.1.

If the region R is a rectangle $\langle a, b\rangle \times \langle c, d\rangle$, i.e.,

$$R = \{[x, y] \in \mathbb{E}_2 : a \leq x \leq b,\ c \leq y \leq d\},$$

we can subdivide the interval $\langle a, b\rangle$ into small intervals using a set of numbers $\{x_0, x_1, \ldots, x_m\}$ so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b.$$

Similarly, a set of numbers $\{y_0, y_1, \ldots, y_n\}$ is said to be a partition of the interval $\langle c, d\rangle$ along the y-axis, if

$$c = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = d.$$

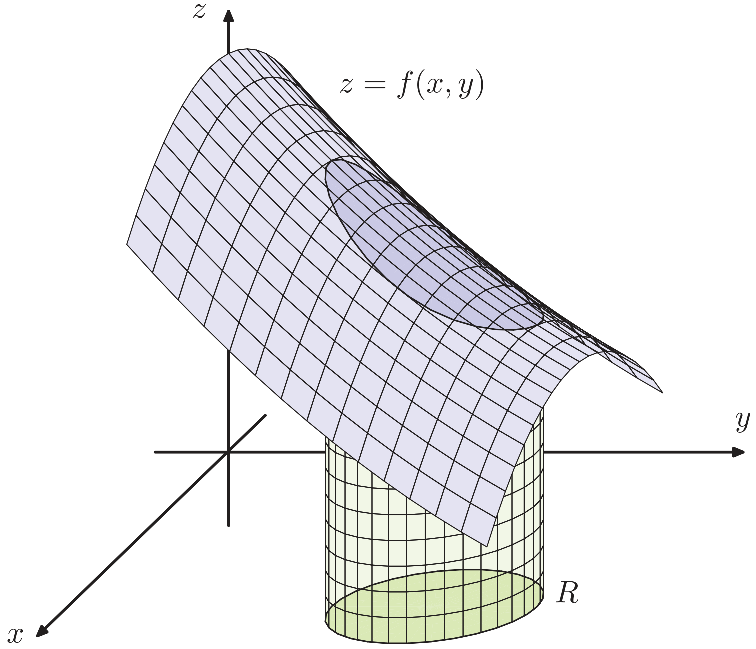

Figure 1.1: Notion of double integral.

If $[x_i^*, y_j^*]$ is some point in the rectangle $\langle x_{i-1}, x_i\rangle \times \langle y_{j-1}, y_j\rangle$ and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ then the *Riemann sum* of a function $f(x, y)$ over the partition of $\langle a, b\rangle \times \langle c, d\rangle$, see Figure 1.2, is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j. \tag{1.1}$$

Then we define the double integral of a function $f(x, y)$ in the rectangular region $\langle a, b\rangle \times \langle c, d\rangle$ as the limit of the Riemann sum as the maximum values of Δx_i and Δy_j approach zero:

$$\iint_{\langle a,b\rangle\times\langle c,d\rangle} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j. \tag{1.2}$$

If the limit in (1.2) exists we say that the function $f(x, y)$ is *integrable* on the region R. The following theorems tell us how to compute a double integral over a rectangle.

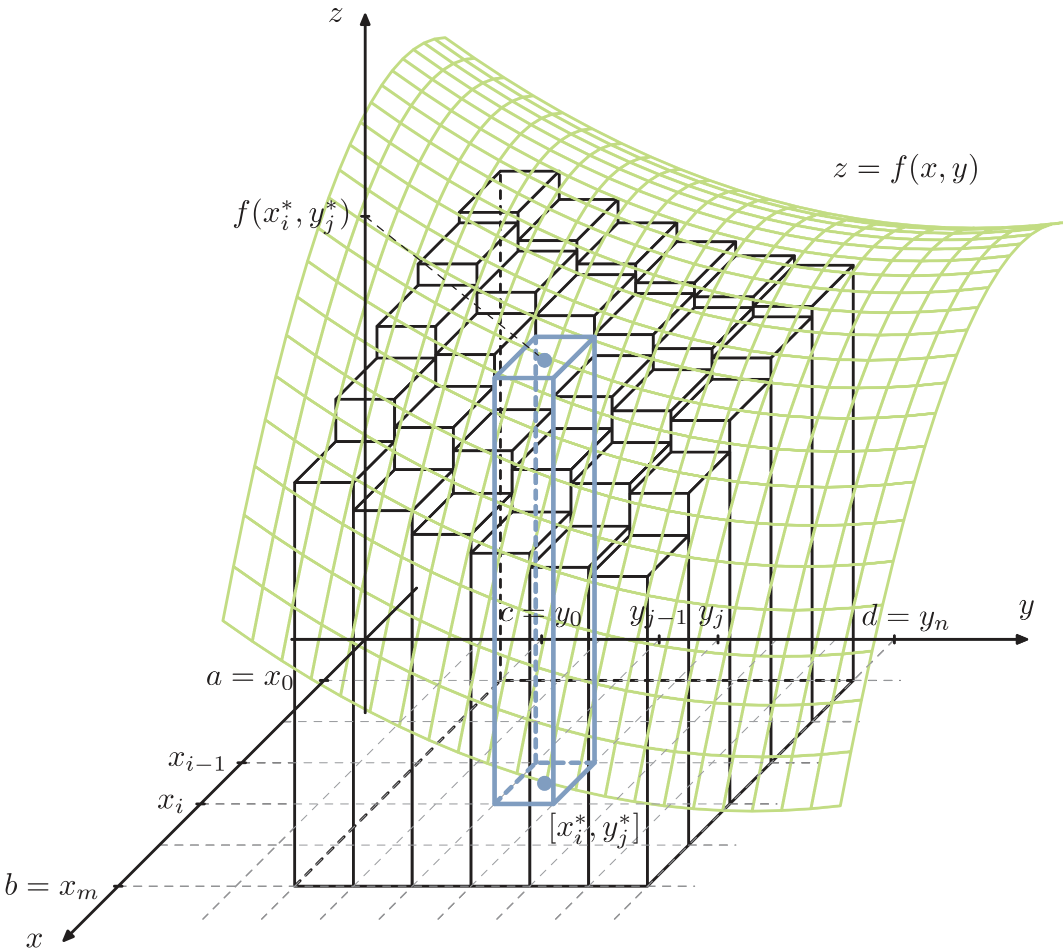

Figure 1.2: Riemann sum of a function $f(x, y)$ over the partition of $\langle a, b\rangle \times \langle c, d\rangle$.

Theorem 1.1.1. *(Fubini's Theorem)*
Let the function f be integrable on a rectangle $R = \langle a, b\rangle \times \langle c, d\rangle$. Then

$$\iint_R f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \int_c^d f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_c^d \int_a^b f(x, y)\, \mathrm{d}y\, \mathrm{d}x.$$

Theorem 1.1.2.
If the integrand $f(x,y)$ is integrable on a rectangle $R = \langle a,b\rangle \times \langle c,d\rangle$ and can be written as a multiplication of two functions each of one variable $f(x,y) = f_1(x)\cdot f_2(y)$, then

$$\iint\limits_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \left(\int\limits_a^b f_1(x)\ \mathrm{d}x\right)\cdot\left(\int\limits_c^d f_2(y)\ \mathrm{d}y\right). \tag{1.3}$$

Example 1.1.1. Compute the double integral $\iint_R 6xy^2\ \mathrm{d}x\ \mathrm{d}y$ over the rectangle $R = \langle 2,4\rangle \times \langle 1,2\rangle$.

Solution In this case we will integrate with respect to y first. Since the $\mathrm{d}y$ is the inner differential, the inner integral needs to have y limits. When we compute the inner integral, we typically keep the outer integral around as follows,

$$\iint\limits_R 6xy^2\ \mathrm{d}x\ \mathrm{d}y = \int\limits_2^4 \left[2xy^3\right]_1^2\ \mathrm{d}x = \int\limits_2^4 14x\ \mathrm{d}x.$$

Remember that we treat x as a constant when doing the first integral and we do not do any integration by x yet. Now, we have a regular integral in one variable and we finish the computation as follows:

$$\iint\limits_R 6xy^2\ \mathrm{d}x\ \mathrm{d}y = \int\limits_2^4 14x\ \mathrm{d}x = \left[7x^2\right]_2^4 = 84.$$

•

If region R is an arbitrary closed region then we have the following theorem.

Theorem 1.1.3.
If f is a continuous function on the closed region R, then $\iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y$ exists.

1.2 Properties of double integrals

In this section we give a survey of general properties of a double integral. These properties are very useful during the calculation of double integrals.

Theorem 1.2.1. *(Homogeneous property)*
Suppose that the function f is integrable over a closed region R and k is an arbitrary constant. Then kf is integrable over the region R and

$$\iint_R kf(x,y)\ \mathrm{d}x\ \mathrm{d}y = k \iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y. \tag{1.4}$$

Theorem 1.2.2. *(Additive property)*
Suppose that the functions f and g are integrable over a closed region R. Then $f+g$ is integrable over the closed region R and

$$\iint_R \big(f(x,y)+g(x,y)\big)\ \mathrm{d}x\ \mathrm{d}y = \iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y + \iint_R g(x,y)\ \mathrm{d}x\ \mathrm{d}y. \tag{1.5}$$

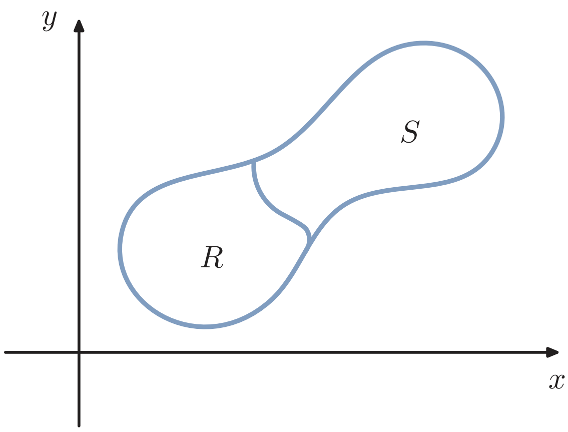

Figure 1.3: Two non-overlapping regions.

Of course, a similar statement is true for the difference of two functions. Consider two closed regions R and S. Figure 1.3 illustrates the location of two non-overlapping regions assumed in Theorem 1.2.3 and Figure 1.4 shows a relationship between regions S and R assumed in Theorem 1.2.4.

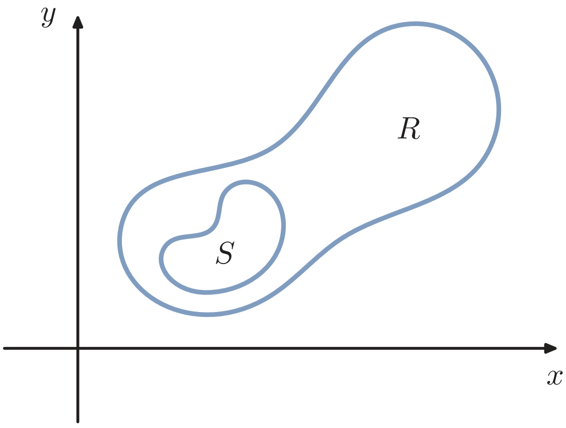

Figure 1.4: Region S is a subregion of R.

Theorem 1.2.3. *(Additivity)*
Let R and S be non-overlapping closed regions and assume that a function f is integrable over the region $R \cup S$. Then

$$\iint_{R\cup S} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \iint_{R} f(x,y)\ \mathrm{d}x\ \mathrm{d}y + \iint_{S} f(x,y)\ \mathrm{d}x\ \mathrm{d}y. \tag{1.6}$$

Theorem 1.2.4.
Suppose that function f is integrable over closed region R and suppose that S is a closed subregion of R. Then

$$\iint_{S} f(x,y)\ \mathrm{d}x\ \mathrm{d}y \leq \iint_{R} f(x,y)\ \mathrm{d}x\ \mathrm{d}y. \tag{1.7}$$

Theorem 1.2.5. *(Non-negativity of the double integral)*
Suppose that function f is integrable over closed region R and let $f(x,y) \geq 0$ over R. Then

$$\iint_{R} f(x,y)\ \mathrm{d}x\ \mathrm{d}y \geq 0. \tag{1.8}$$

Theorem 1.2.6. *(Monotone property of the double integral)*
Suppose that functions f and g are integrable over closed region R with $g(x,y) \leq f(x,y)$, for all $[x,y] \in R$. Then

$$\iint_R g(x,y)\ \mathrm{d}x\ \mathrm{d}y \leq \iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y. \tag{1.9}$$

Theorem 1.2.7. *(Median of the double integral)*
If f is a continuous function on closed region R and $A(R)$ is the area of R, then there exists at least one point $[x_i, y_j] \in R$ such that

$$\iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y = f(x_i, y_j) \cdot A(R). \tag{1.10}$$

1.3 Iterated integrals

In Section 1.1 we looked at double integrals over rectangular regions. However, most of the regions are not rectangular so we need now to look at the following double integral

$$\iint_D f(x,y)\ \mathrm{d}x\ \mathrm{d}y,$$

where D is an arbitrary region. There are two types of regions that need to be considered.

Definition 1.3.1. *(Normal domain with respect to the x-axis)*
The normal domain with respect to the x-axis is bounded by lines $x = a$ and $x = b$, where $a < b$, and continuous curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$, where $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in \langle a, b\rangle$.

Let D be a region lying between the curves of φ_1 and φ_2 over an interval $\langle a, b\rangle$, see Figure 1.5. Thus

$$D = \{[x,y] \in \mathbb{E}_2 : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}. \tag{1.11}$$

Then the area between the curves of $\varphi_1(x)$ and $\varphi_2(x)$ from a to b is

$$A(D) = \int_a^b \big(\varphi_2(x) - \varphi_1(x)\big)\, \mathrm{d}x. \tag{1.12}$$

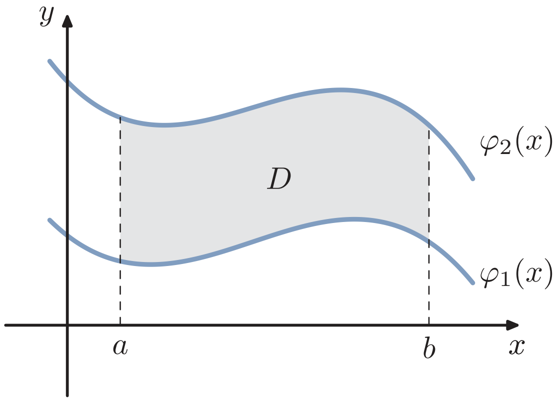

Figure 1.5: Definition of a region D.

Definition 1.3.2. *(Normal domain with respect to the y-axis)*
The normal domain with respect to the y-axis is bounded by lines $y = c$ and $y = d$, where $c < d$, and continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$, where $\psi_1(y) \leq \psi_2(y)$ for all $y \in \langle c, d\rangle$.

Let G be a region lying between the curves of ψ_1 and ψ_2 over an interval $\langle c, d\rangle$, see Figure 1.6. Thus

$$G = \{[x, y] \in \mathbb{E}_2 : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}. \tag{1.13}$$

Then the area between the curves of $\psi_1(y)$ and $\psi_2(y)$ from c to d is

$$A(G) = \int_c^d \big(\psi_2(y) - \psi_1(y)\big)\, \mathrm{d}y. \tag{1.14}$$

The following theorems are stronger form of Fubini's Theorem and state how to express a double integral in terms of iterated integrals.

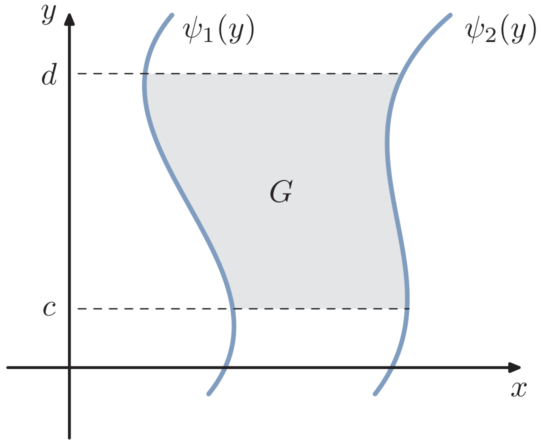

Figure 1.6: Definition of a region G.

Theorem 1.3.1.
Let $f(x,y)$ be a continuous function on the closed region $D = \{[x,y] \in \mathbb{E}_2 : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$ and let $\varphi_1(x)$, $\varphi_2(x)$ be continuous functions defined on the interval $\langle a,b\rangle$ with $\varphi_1(x) \le \varphi_2(x)$, for all x in $\langle a,b\rangle$. Then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x. \tag{1.15}$$

Theorem 1.3.2.
Let $f(x,y)$ be a continuous function on the closed region $G = \{[x,y] \in \mathbb{E}_2 : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$ and let $\psi_1(y)$, $\psi_2(y)$ be continuous functions defined on the interval $\langle c,d\rangle$ with $\psi_1(y) \le x \le \psi_2(y)$, for all y in $\langle c,d\rangle$. Then

$$\iint_G f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y. \tag{1.16}$$

Let's take a look at some examples of double integrals over general regions.

Example 1.3.1. Evaluate the integral

$$\iint_D (x^2 + y)\ \mathrm{d}x\ \mathrm{d}y,$$

where $D = \{[x, y] \in \mathbb{E}_2 : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$.

Solution According to Theorem 1.3.1 we get

$$\iint_D (x^2 + y)\ \mathrm{d}x\ \mathrm{d}y = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y)\ \mathrm{d}x\ \mathrm{d}y = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (x^2 + y)\ \mathrm{d}y \right]\ \mathrm{d}x$$

$$= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}}\ \mathrm{d}x = \int_0^1 \left(x^2\ \sqrt{x} + \frac{x}{2} - \frac{3x^4}{2} \right)\ \mathrm{d}x$$

$$= \left[\frac{2\sqrt{x^7}}{7} + \frac{x^2}{4} - \frac{3x^5}{10} \right]_0^1 = \frac{33}{140}.$$

•

Example 1.3.2. Evaluate the integral

$$\iint_G \frac{x^2}{y^2}\ \mathrm{d}x\ \mathrm{d}y,$$

where $G = \{[x, y] \in \mathbb{E}_2 : 1 \le y \le 2,\ 1/y \le x \le y\}$.

Solution According to Theorem 1.3.2 we get

$$\iint_G \frac{x^2}{y^2}\ \mathrm{d}x\ \mathrm{d}y = \int_1^2 \int_{\frac{1}{y}}^{y} \frac{x^2}{y^2}\ \mathrm{d}y\ \mathrm{d}x = \int_1^2 \left[\int_{\frac{1}{y}}^{y} \frac{x^2}{y^2}\ \mathrm{d}x \right]\ \mathrm{d}y = \frac{1}{3} \int_1^2 \left[\frac{x^3}{y^2} \right]_{\frac{1}{y}}^{y}\ \mathrm{d}y$$

$$= \frac{1}{3} \int_1^2 \left(y - \frac{1}{y^5} \right)\ \mathrm{d}y = \frac{1}{3} \left[\frac{y^2}{2} + \frac{1}{4y^4} \right]_1^2 = \frac{27}{64}.$$

•

The region $D = \{[x, y] \in \mathbb{E}_2 : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$ considered in Example 1.3.1 is depicted in Figure 1.7 and the region $G = \{[x, y] \in \mathbb{E}_2 : 1 \le y \le 2,\ 1/y \le x \le y\}$ considered in Example 1.3.2 is illustrated in Figure 1.8.

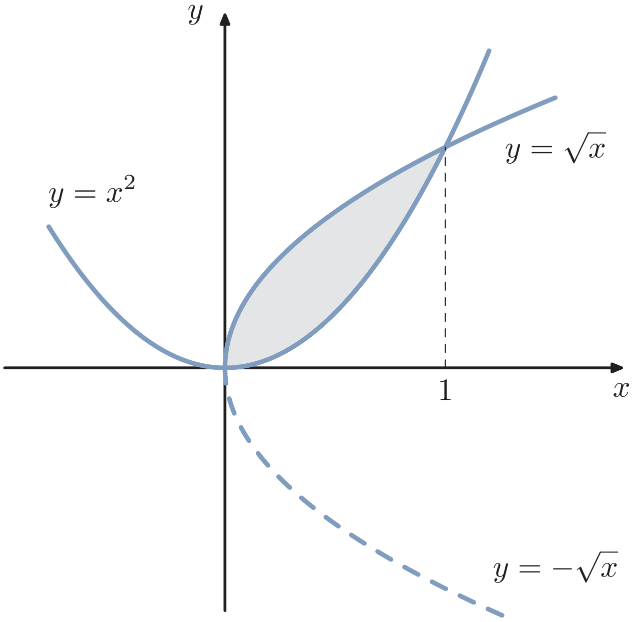

Figure 1.7: The region $D = \{[x, y] \in \mathbb{E}_2 : 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\}$.

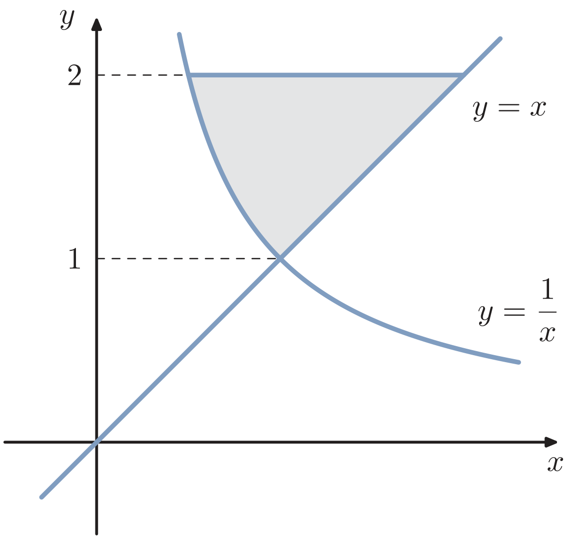

Figure 1.8: $G = \{[x, y] \in \mathbb{E}_2 : 1 \leq y \leq 2,\ \frac{1}{y} \leq x \leq y\}$.

Example 1.3.3. Evaluate the integral $\iint_D x^2 y \, \mathrm{d}x \, \mathrm{d}y$ over the region D bounded above by the line $y = 6 - x$ and below by the curve $y = 5/x$.

Solution To find the region D, we need to know the bounds where the region begins and ends. Therefore, we sketch the curve and line together in the same graph, see Figure 1.9.

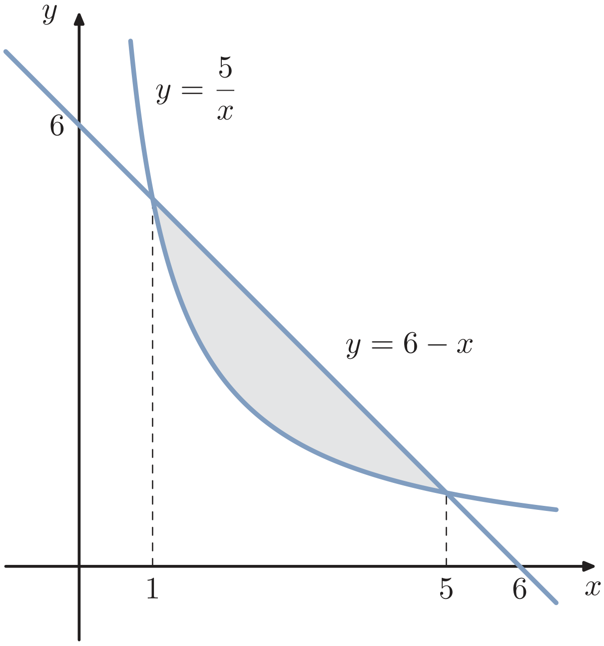

Figure 1.9: The region bounded by the line $y = 6 - x$ and the curve $y = 5/x$.

To find the points of intersection of two curves, we set

$$\frac{5}{x} = 6 - x$$

and solve for x

$$\begin{aligned} \frac{5}{x} - 6 + x &= 0, \\ \frac{5 - 6x + x^2}{x} &= 0, \\ (x-5)(x-1) &= 0. \end{aligned}$$

Hence $x = 1$ and $x = 5$ are the x-coordinates of two intersections. Since

$$\frac{5}{x} \leq 6 - x$$