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SYSTEM AS A BASIC TOOL IN SOLUTION OF TASKS ON THE BASIS OF SYSTEM SCIENCE KNOWLEDGE BASE

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Abstract: *Solutions to most complex tasks in the area of technology, economics, control, sociology, as well as tasks of scientific research in these areas, rely upon the knowledge of the system science. The system science as a separate field of science took shape at the time of scientific and technical revolution.*
It includes several scientific disciplines of which the theory of systems has the most comprehensive character. In this contribution we point out the relationship between the linear tasks in the abstract sense and vector spaces as abstract mathematical structures. At the same time we try to maintain system approach. This way the common basis of solution of superficially different tasks is emphasized.

Key words: *basic tool in solution*

1 INTRODUCTION

Theory of systems (Bertalanffy, 1950) is a generalized theory of modeling the system properties of an object. It is used as a fundamental methodological tool in other disciplines of the system science, but also generally in the solution of particular tasks of science and practice. We speak about the system approach toward the solution of the task. It consists in the definition of a real system encompassing the object or process being solved. Under the notion of a system we understand an ordered pair $S=(A,R)$, where A is the set of elements of the system, $R \subseteq A \times A$ is the relation between them.

Theory of systems [1] provides several general systems having mathematical structure and represent type models of relations between the elements (variables) of such abstract systems. At the same time methods of solution of these general systems are available in the sense of explicit expression of unknown variables as functions of known variables and parameters of the system.

The system approach towards the solution of a task, which consists in the definition of a real system over the given problem, requires using abstraction and mathematical formalism as a tool. By suitable application of these tools we strip the problem being solved of unnecessary conditions and gradually its structure will appear, corresponding to the elements and the relations among them. By comparing the obtained structure of the real system with the set of general type systems which the theory offers, a corresponding general system will be determined and also the method of solution of real system as well as the original task will follow.

The majority of solved practical tasks has the structure of linear task and boils down to some of these linear general systems:

- general system (set) of homogeneous and nonhomogeneous linear algebraic equations with n unknowns,
- general system (set) of homogeneous and nonhomogeneous ordinary linear differential equations with constant coefficients,
- general system (set) of homogeneous and nonhomogeneous partial linear differential equations with constant coefficients,
- linear integral equations and their systems,
- linear transformations.

All linear tasks can be looked upon via the structure of *general system of linear algebraic equations*, known from elementary algebra:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = y_m \quad (1)$$

System of equations (1) represents general system of linear algebraic equations. The above definition of the system in this case takes on a more specific form of ordered pair $S=(A,R)$, where the set of elements of the system $A=\{X_j\}_{j=1}^{n} \cup \{Y_i\}_{i=1}^{m}$, R is the set (system) of relations $R=\{r_i\}_{i=1}^{m}$, while $r_i \subset \left(\prod_{j=1}^{n} X_j\right) \times Y_i$. The symbol $X_j, j=1,2,\ldots,n$ denotes the domain of possible values of unknown variable x_j and the symbol $Y_j, j=1,2,\ldots,m$ denotes the domain of possible values of variable y_i as a restriction. So in this case the elements of the system are the quantities $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_m$, or their domains of values. Relation R of the system has the form of system of equations (1).

2 THE TASK OF LINEAR PROGRAMMING AS A MODEL OF SOLUTION OF LINEAR TASKS AND ITS CONNECTION WITH VECTOR SPACES

The fundamental meaning of general system of linear algebraic equations (1) for logistics is contained in extended *task of linear programming.* It is a model for the solution of transportation task, allocation problem, distribution problem and other linear tasks.

Linear programming generally consists in the task of finding such a real number $x_1, x_2, ..., x_n$ (quantified decisions) for which $c_1x_1 + c_2x_2 + ... + c_nx_n$ at least under the assumption that equalities (1) hold (Malindžák D. and al., 2009). From the application point of view it is about finding the optimum production program $\mathbf{x}_0 = (x_{01}, x_{02}, ..., x_{0n})$ as an algebraic vector from the set of all possible plans $\mathbf{x} = (x_1, x_2, ..., x_n)$, whose *i*-th component $x_{0i} \geq 0$ represents the planned amount of *i*-th kind of product.

The objective function $f(\mathbf{x}) = c_1x_1 + c_2x_2 + ... + c_nx_n = (\mathbf{c}, \mathbf{x})$ expresses the measure of usefulness of the realization of production plan **x**. Here the vector **c** contains the coefficients of usefulness of unit amount of individual products. The creation of products requires resources whose capacity is limited. The consumption amounts of products are defined and limited by consumers. The influence of these limitations from the resources and consumers side on the production plan is expressed by a system of equations or inequalities of type (1). Here the matrix $(a_{ij})_{m \times n}$ of coefficients of the system represents specific consumption of *i*-th resource on *j*-th product. The right-hand side of the system is represented by restrictions of the resources or consumers. So the task of linear programming is to maximize (minimize) the objective function $f(\mathbf{x})$ while satisfying restrictions in the form of system of linear algebraic equations (1) which leads to finding optimum argument $\mathbf{x}_0$ (optimum production plan) of this objective function. In *Fig.1* is shown the structure of the task of linear programming as a linear static system.

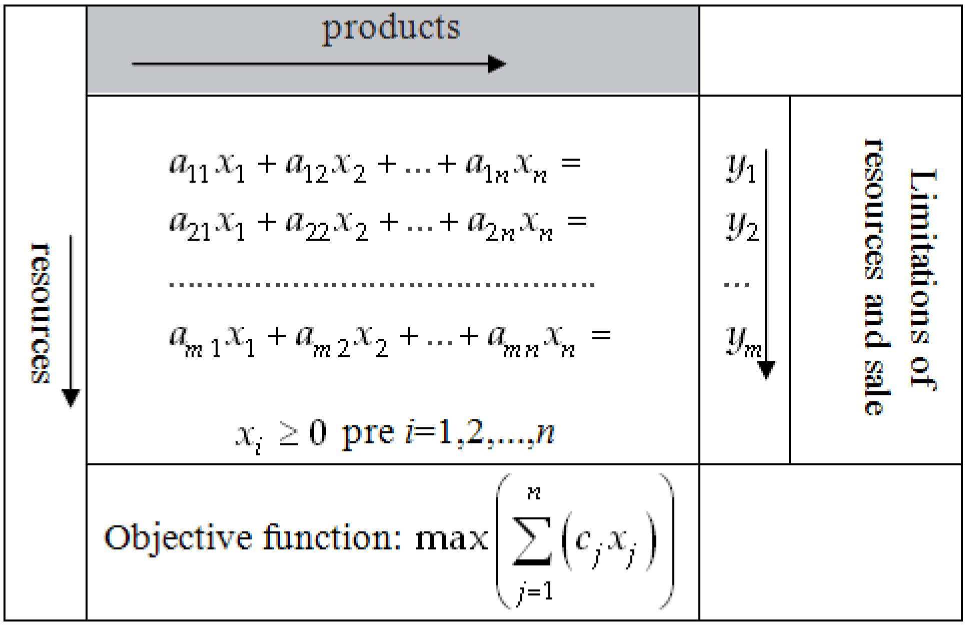

Fig. 1 Structure of the task of linear programming

Interesting is the view of the task of linear programming via vector spaces. That is because all linear tasks are in general characterized by being directly associated with *linear vector spaces.* It follows from the structure in *Fig.1* that the task of linear programming is based upon the mapping the finite-dimensional normed euclidean space $X = V_n(\square) \equiv E_n$, containing all possible production plans $\{\mathbf{x}_i\}_i$, into the finite-dimensional normed euclidean space $Y = V_m(\square) \equiv E_m$ containing all possible vectors of restrictions $\{\mathbf{y}_i\}_i$.

If the euclidean orthonormal base of the space X consists of vectors $\{\mathbf{e}_k\}_{k=1}^{n}$, where $\mathbf{e}_1 = (1,0,\ldots,0)$,..., $\mathbf{e}_n = (0,0,\ldots,1)$, then each production plan as a vector $\mathbf{x} \in X$ can be expanded relative to this base into the series $\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{e}_k$. This series represents a linear combination of planned amounts of individual products and corresponding base vectors that represent units in which the planned amounts of products are expressed. This way each possible production plan corresponds to unique point - vector in the space *X*. At the same time a possibility appears to use euclidean metric for the calculation of „distances“ (differences) of individual variants of the production plan.

If analogously the orthonormal base of the space Y is formed by vectors $\{\mathbf{f}_k\}_{k=1}^{m}$, where $\mathbf{f}_1 = (1,0,\ldots,0)$,..., $\mathbf{f}_m = (0,0,\ldots,1)$, then each vector of restrictions $\mathbf{y} \in Y$ can be expanded relative to this base into the series $\mathbf{y} = \sum_{k=1}^{m} y_k \mathbf{f}_k$. This series represents a linear combination of individual restrictions and corresponding base vectors which represent units in which these restrictions are expressed.

The above mapping of the space of production plans X into the space of restrictions Y is realizedby a continuous linear operator $\hat{A}$ defined on X with the values in Y, which generally realizes the mapping of the linear normed space E_n into the linear normed space E_m. This operator is uniquely determined by the matrix $\mathbf{A} = \left(a_{ij}\right)_{m \times n}$ of system (1) and the mapping V realized by it can be expressed with the vector equation:

$$\hat{A}(\mathbf{x}) = \mathbf{A}\mathbf{x}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}} \tag{2}$$

So in general, all linear tasks based upon a general system of linear algebraic equations (1) are related to this linear operator $\hat{A}$. Its properties can be determined only by investigating the matrix $\mathbf{A}$ (Taylor, 1967), (Dorf, Bishop, 1990).

One of the properties of continuous linear operator $\hat{A}$ mapping E_n into E_m is the fact that in the space E_m there always exist n vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ such that for all $\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right) \in E_n$ we have

$$\hat{A}\left(\mathbf{x} \in E_n\right) = \sum_{k=1}^{n} x_k \mathbf{b}_k \equiv \mathbf{y} \in E_m \tag{3}$$

In other words by the linear opreator $\hat{A}$ a vector $\mathbf{x} \in E_n$ is mapped onto the vector $\mathbf{y} \in E_m$, which in the space E_m (or its subspace) , relative to a particular existing base $\{\mathbf{b}_k\}_{k=1}^{n}$ of this space, has the same coordinates as the original vector x in E_n. In this context a question arises about the method of determining such a base $\{\mathbf{b}_k\}_{k=1}^{n}$ of the space E_m.

Let us apply the above linear mapping $X \equiv E_n \rightarrow Y \equiv E_m$, represented by the operator $\hat{A}$, on the vector $\mathbf{x} \in E_n$. We get

$$\hat{A}(\mathbf{x}) = \hat{A}\left(\sum_{k=1}^{n} x_k \mathbf{e}_k\right) = \sum_{k=1}^{n}\left[x_k \hat{A}(\mathbf{e}_k)\right] \tag{4}$$

where, in the sense of superposition, we used additivity and homogeneity of this linear operator. By applying the operator $\hat{A}$ on the euclidean base vector $\mathbf{e}_k$ in (4) we get

$$\hat{A}(\mathbf{e}_k) = \mathbf{A}\mathbf{e}_k^{\mathrm{T}} = (a_{1k}, a_{2k}, \ldots, a_{mk})^{\mathrm{T}} = \mathbf{a}_k \tag{5}$$

We see that the result of action of the linear operator $\hat{A}$ on the base vector $\mathbf{e}_k$ of the orthonormal base of the space E_n is k-th column $\mathbf{a}_k$ of matrix **A** of this operator which becomes vector of the space E_m. This vector can be expanded relative to the orthonormal base $\{\mathbf{f}_k\}_{k=1}^{m}$ into Fourier series

$$\mathbf{a}_k = (a_{1k}, a_{2k}, \ldots, a_{mk})^{\mathrm{T}} = \sum_{i=1}^{m} a_{ik}\mathbf{f}_i^{\mathrm{T}} \tag{6}$$

If in relation (6), for the expression of j-th coordinate of vector $\mathbf{f}_i$, we use the Kronecker symbol δ_{kj}, where $\delta_{kj} = 1$ for $k = j$ and $\delta_{kj} = 0$ for $k \neq j$, we obtain

$$\mathbf{a}_k = \sum_{i=1}^{m} a_{ik}\mathbf{f}_i^{\mathrm{T}} = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\delta_{kj}\right)\mathbf{f}_i^{\mathrm{T}} \tag{7}$$

Successive substitution of (5), (6) and (7) into (4) yields

$$\hat{A}(\mathbf{x}) = \hat{A}\left(\sum_{k=1}^{n} x_k \mathbf{e}_k\right) = \sum_{k=1}^{n}\left[x_k \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\delta_{kj}\right)\mathbf{f}_i^{\mathrm{T}}\right] =$$

$$= \sum_{k=1}^{n}\left[x_k \mathbf{a}_k\right] = \mathbf{y} = (y_1, y_2, \ldots, y_m). \tag{8}$$

The derived operator expression (8) represents other form of expressing the system of linear algebraic equations (1). It follows from it that to solve the system (1) means to find the coordinates $x_1, x_2, ..., x_k, ..., x_n$ of vector $\mathbf{y} \in Y \equiv E_m$ in the base $\{\mathbf{a}_k\}_{k=1}^{n}$, when the base vectors $\mathbf{a}_k \in Y \subset E_m$ for k=1,2,...,n are the columns of the matrix **A** of coefficients of the system. These unknown coordinates of vector **y** are the coefficients of Fourier expansion of vector **y** using base $\{\mathbf{a}_k\}_{k=1}^{n}$ in the space Y. From the viewpoint of the task of linear programming it is about finding such a production program x whose coordinates as a vector in E_n relative to the orthonormal euclidean base are at the same time the coordinates of the vector of restrictions y in the subspace of the space E_m relative to the system of vectors as columns of transformation matrix A of the coefficients of specific expenses. At the same time, given objective function must be satisfied.

3 LINEAR PROBLEMS IN HILBERT SPACES

We call Hilbert space a space that is infinite-dimensional, complex, linear vector space $\square^{\infty}$, which is complete and separable, and there is an inner product defined on the space (Taylor, 1967), (Vagn, 2006), (Leššo, Flegner, Feriančiková, 2011)4. The subject of these abstract mathematical spaces is dealt with in functional analysis. It should be noted at this point that the elements of a Hilbert space can be functions as vectors satisfying certain criteria. There are several classes of Hilbert spaces that differ in their set structure i.e. in the definition of the functions as elements of the space.

Let us consider a class l_p, $1 \le p < \infty$ of Hilbert spaces, where the elements of the space are all infinite sequences $\{x_k\}_{k=1}^{\infty}$ of complex or real numbers such that the series of the terms of sequence $\sum_{k=1}^{\infty} |x_k|^p < \infty$ (thus the series converges). Such a sequence of numbers can be regarded as function or vector of Hilbert space with infinite number of complex or real coordinates, so that the vector $x = (x_1, x_2, ...) \in l_p \equiv \square^{\infty}$. A standard convention is used here that vectors of Hilbert space are not written in bold letters as with euclidean spaces. For l_p the space are defined algebraic operations, analogous to the operations in finite-dimensional spaces of euclidean type.

In Hilbert space it is possible to define so-called complete system of mutually orthonormal vectors $\{b_k\}_{k=1}^{\infty}$, where for each couple of vectors of this system $(b_k, b_l) = \delta_{kl}$. Then for each function x as a point of the Hilbert space (analogously to the euclidean space E_n with euclidean base) can be expanded relative to the given system of orthonormal vectors into Fourier series

$$x = \sum_{k=1}^{\infty} x_k b_k \quad . \qquad (9)$$

The coefficients x_k, k=1,2,.. , of the above Fourier series (9) represent the spectrum of vector x relative to base $\{b_k\}_{k=1}^{\infty}$. In addition, orthonormality or orthogonality of the base makes it possible to calculate the coefficients of the expansion by using inner product

$$x_k = (x, b_k) = \sum_{j=1}^{\infty} x_j^* b_{kj} , \qquad (10)$$

where $x_j, j = 1,2,...$ are the coordinates of the vector in the original base and $x_k, k = 1,2,...$ are new coordinates of the same vector in the orthonormal base $\{b_k\}_{k=1}^{\infty}$.

To illustrate that the Fourier expansion (9) of function x as a vector in Hilbert space also has the structure of general linear system (1), similar to the problem of linear programming (Fig.1), is shown a solution of a particular problem.

Available is a time waveform of air temperature (Fig.2) measured at the same place during four days. The task is to analyze and suitably visualize the dynamics of this quantity. The above problem has a general use, since any complex or real continuous function, signal etc. can be imagined instead of the temperature waveform. It could be, for example, a monitored consumption in logistics of a continuous medium, time series in the area of finance, markets, etc.

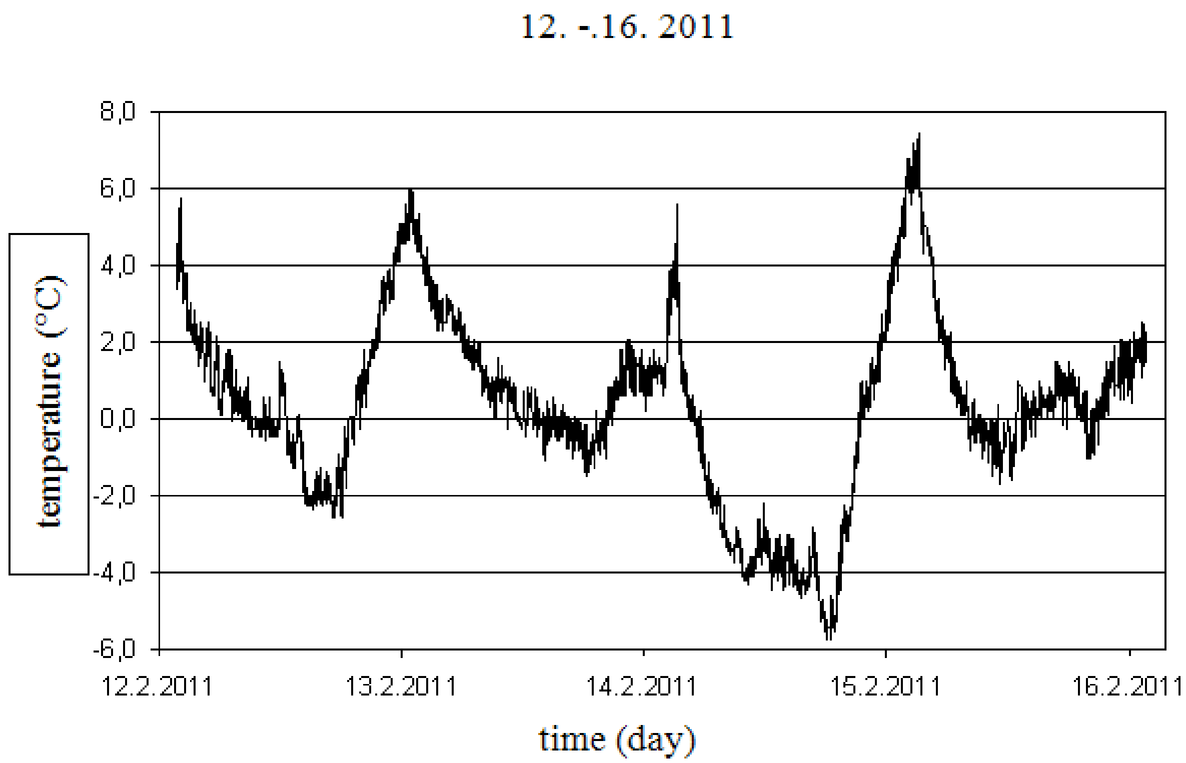

Fig. 2 Temperature waveform as function - vector in Hilbert space

The measured temperature waveform represents, under certain generalization, a function of time $x = x(t)$, $t \in \langle 0, 24\ \text{h} \rangle$. If we formally express it as an infinite sequence of real numbers, we can regard it as a vector of Hilbert space (its real subspace) of class

l_p. Individual measured values of the temperature represent the coordinates of the vector $x = (x_0, x_1, x_2, ..., x_j, ...)$ in orthonormal base $\{b_k\}_{k=0}^{\infty}$, where $b_k = \{\delta_{kj}\}_{j=0}^{\infty}$ for k=0,1,2,... . At the same time the base vector for k=0 corresponds to unidirectional component (ω =0), index j corresponds to time coordinate. For the purpose of discovering or confirming the expected periodicity in the behavior of the temperature, the function, as a vector $x \in l_p$, was expanded in Hilbert space into a Fourier series (9) in complex orthogonal base of harmonic functions in exponential form $\{b_k\}_{k=0}^{\infty}$, where we have for the base vector $b_k = \left(\mathrm{e}^{\mathbf{i}k\Delta\omega 0 T_{\mathrm{vz}}}, \mathrm{e}^{\mathbf{i}k\Delta\omega 1 T_{\mathrm{vz}}}, \mathrm{e}^{\mathbf{i}k\Delta\omega 2 T_{\mathrm{vz}}}, ..., \mathrm{e}^{\mathbf{i}k\Delta\omega j T_{\mathrm{vz}}}, ...\right)$. Minimum time step T_{vz} is assumed, theoretically approaching zero, similarly frequency resolution $\Delta\omega$. The Fourier complex coefficients of this expansion are given by relation (10), divided by constant T - the length of recording, because the harmonic base is not orthonormal, only orthogonal.

A reasoning can lead to the knowledge, that the calculation of all coefficients of the unknown expansion (9) by using scalar product in the form (10) can again be expressed in operator form (2) as was the case in the problem of linear programming. It suffices to consider the matrix of the operator as infinite-dimensional and create their rows from individual base vectors of harmonic functions. So $\mathbf{A} = (a_{kj})_{\infty\times\infty}$, where $a_{kj} = \mathrm{e}^{-\mathbf{i}k\Delta\omega j T_{\mathrm{vz}}}$. The entire expansion of the vector x in Hilbert space of class l_p in orthogonal base is then realized by linear operator

$$\hat{A}\left(x_t^{\mathrm{T}}\right) = \frac{1}{T}\mathbf{A}x_t^{\mathrm{T}} = x_\omega^{\mathrm{T}} \quad . \qquad (11)$$

Then the measured waveform of the temperature, identified with vector x expressed in orthonormal base $b_k = \{\delta_{kj}\}_{j=0}^{\infty}$, k=0,1,2,.. is denoted by symbol x_t. The coordinates of this vector in orthogonal base of harmonic functions as the result of the mapping (11) are expressed as vector x_ω.

Graphically illustrated in *Fig. 3* is the operator equation (11), where the rows of infinite-dimensional matrix **A** of linear operator $\hat{A}$ in Hilbert space l_p are harmonic functions as mutually orthogonal base vectors.

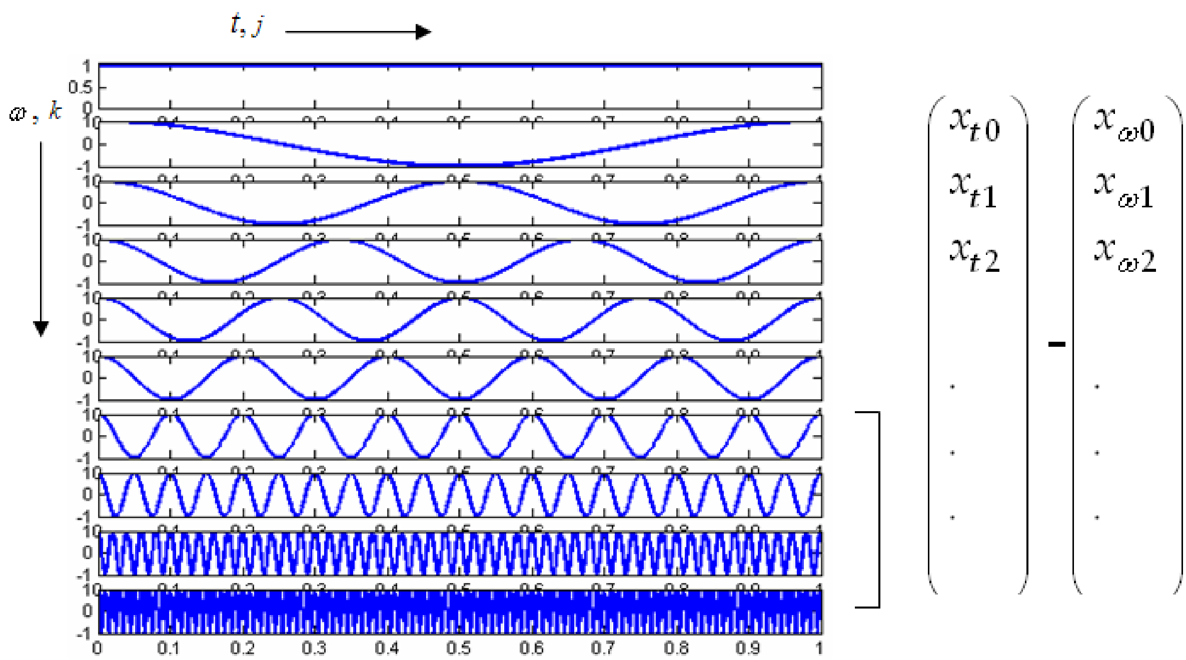

Fig. 3 Graphical illustration of matrix A of the operator in equation (11).

In Fig.4 is shown function $\left|x_{\omega}\right| = \left|x_{\omega}\right|(2\pi/\omega)$ as a result of the expansion of the function $\mathbf{x}_t$ in the base of complex harmonic functions. Two extremes of the function $\left|\mathbf{x}_{\omega}\right|$ confirm in the four-day waveform $\mathbf{x}_t$ periodic repetition of temperatures after 24 hours and less distinctly even after 12 hours. As a matter of fact the figure shows classical amplitude frequency spectrum by noting that in the lower axis, instead of the frequency there is its inverse value i.e. the period.

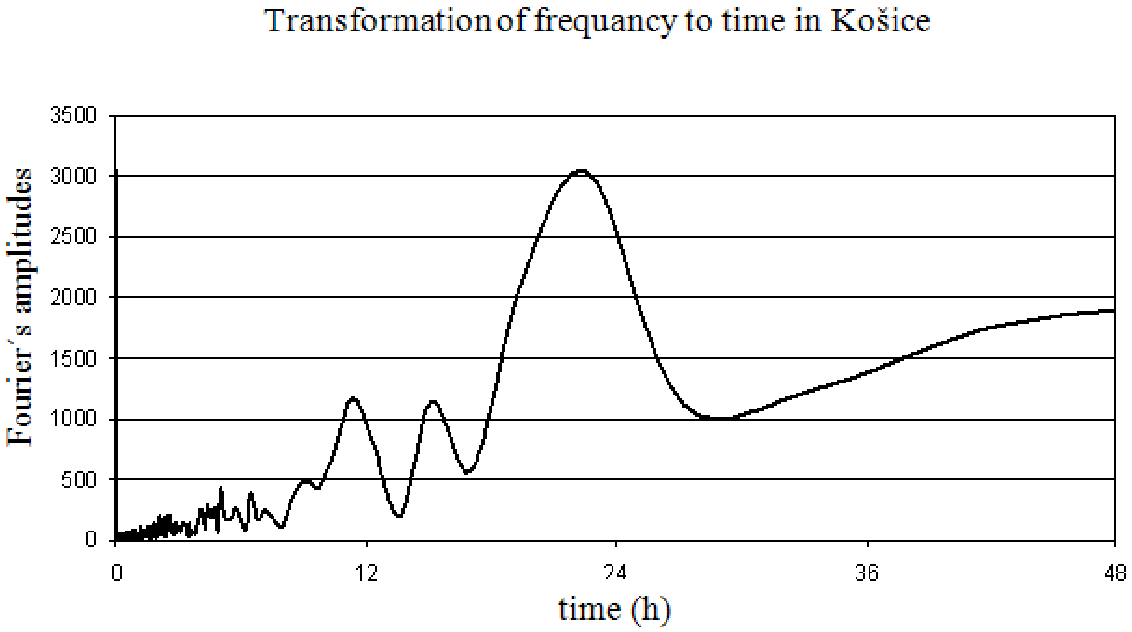

Fig. 4 Waveform of the function as a vector of temperatures expanded in Hilbert space in the base of harmonic functions.

The tools of functional analysis in processing the behavior of functions as vectors in Hilbert space make it possible to use geometric structure of this space for nonstandard visualization of properties of these functions. In the investigation of the dynamics of the behavior of temperatures the following procedure was used:

The behavior of temperatures over one hour was defined as a vector of Hilbert space of class l_p. Let's denote it by symbol $x_{\mathrm{h}t}$. This vector was expanded in complex base of harmonic functions with the above-mentioned linear operator $\hat{A}\left(x_{\mathrm{h}t}\right)=x_{\mathrm{h}\omega}$ according to (9) and (10). Vector $x_{\mathrm{h}\omega}$ represents the complex spectrum of one-hour behavior of the temperature. In the following procedure only amplitude spectrum of this expansion denoted by symbol $\left|x_{\mathrm{h}\omega}\right|$ was monitored.

The dynamic evolution of the measured temperature causes the motion of such defined point – vector $\left|x_{\mathrm{h}\omega}\right|$ in space l_p since each hourly behavior of temperatures represents its new coordinates. For the purposes of visualizing this dynamics the norm of the vector $\left|x_{\mathrm{h}\omega}\right|$ was calculated by using the relation

$$\left\|\left(\left|x_{\mathrm{h}\omega}\right|\right)\right\|=\sqrt{\sum_{k=0}^{\infty}\left|x_{\mathrm{h}\omega\,k}\right|^{2}}, \qquad (12)$$

which is a real number and can be considered a measure of heat energy at the place of measurement. Further, a phase plane for the observation of the dynamics of the vector

$\left|x_{h\omega}\right|$ was defined. It is the plane $\left(\left\|\left(\left|x_{h\omega}\right|\right)\right\|, \operatorname{grad}\left(\left\|\left(\left|x_{h\omega}\right|\right)\right\|\right)\right)$. The result of this procedure is shown in Fig.5. It is a trajectory of the vector $\left|x_{h\omega}\right|$ during the period of 24 hours. The figure illustrates in an interesting manner the fact that the temperature of the outer environment under current climate conditions is a chaotic cyclic attractor.

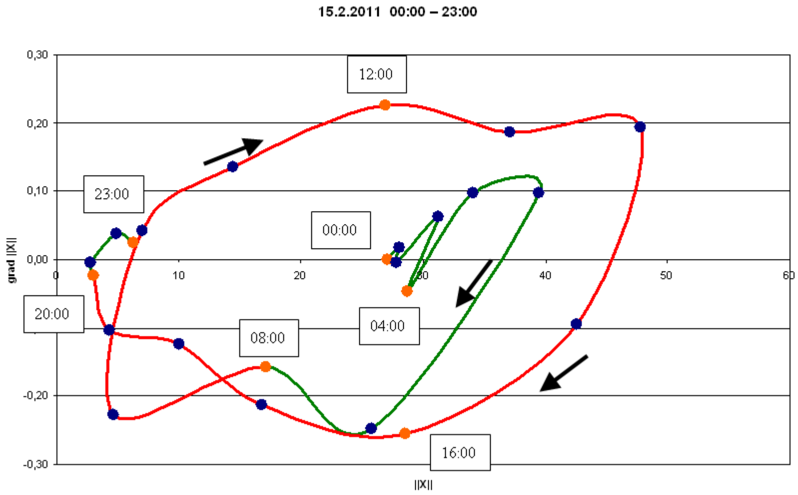

Fig. 5 Suitable interpretation of the measured behavior of air temperature as a vector in Hilbert space enables to exactly prove its character of cyclic chaos.

6 CONCLUSIONS

In this contribution the fact is pointed out that all linear problems possess the structure originating from a general linear system having the form of system of linear equations. These linear problems are directly related to abstract linear vector spaces. It is shown that, for example, the solution of the problem of linear programming in logistics has the same formal structure as the problem of harmonic analysis of signals in the area of digital signal processing.

For illustration purposes an application is given of infinite-dimensional Hilbert space to analysis of dynamical properties of external air temperature in given location of measurement. The results exhibit chaotic cyclic orbits of such nonlinear dynamical system.

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