Shafts

Introduction

A shaft is a rotating member, usually of circular cross section, used to transmit power or motion. It provides the axis of rotation, or oscillation, of elements such as gears, pulleys, flywheels, cranks, sprockets, and the like and controls the geometry of their motion. An axle is a nonrotating member that carries no torque and is used to support rotating wheels, pulleys, and the like. The automotive axle is not a true axle; the term is a carryover from the horse-and-buggy era, when the wheels rotated on nonrotating members. A nonrotating axle can readily be designed and analyzed as a static beam, and will not warrant the special attention given in this chapter to the rotating shafts which are subject to fatigue loading.

Shaft Design for Stress

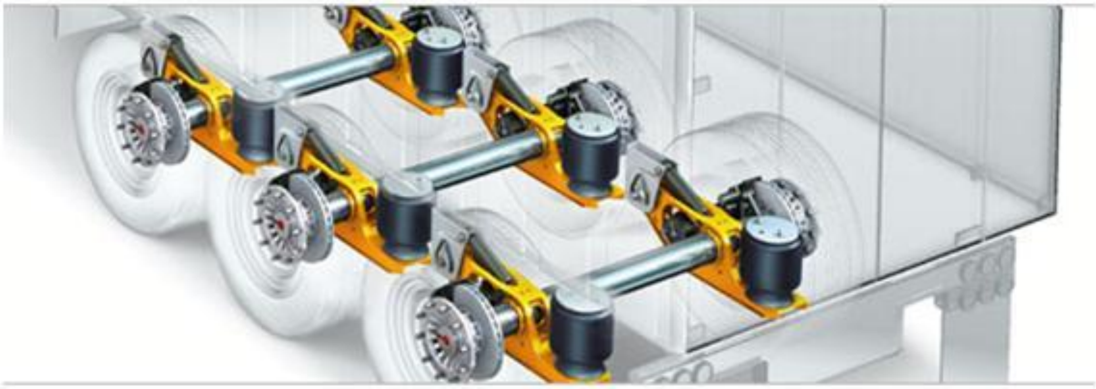
It is not necessary to evaluate the stresses in a shaft at every point; a few potentially critical locations will suffice. Critical locations will usually be on the outer surface, at axial locations where the bending moment is large, where the torque is present, and where stress concentrations exist. By direct comparison of various points along the shaft, a few critical locations can be identified upon which to base the design. An assessment of typical stress situations will help.

Most shafts will transmit torque through a portion of the shaft. Typically the torque comes into the shaft at one gear and leaves the shaft at another gear. A free body diagram of the shaft will allow the torque at any section to be determined. The torque is often relatively constant at steady state operation. The shear stress due to the torsion will be greatest on outer surfaces.

The bending moments on a shaft can be determined by shear and bending moment diagrams. Since most shaft problems incorporate gears or pulleys that introduce forces in two planes, the shear and bending moment diagrams will generally be needed in two planes. Resultant moments are obtained by summing moments as vectors at points of interest along the shaft. The phase angle of the moments is not important since the shaft rotates. A steady bending moment will produce a completely reversed moment on a rotating shaft, as a specific stress element will alternate from compression to tension in every revolution of the shaft. The normal stress due to bending moments will be greatest on the outer surfaces. In situations where a bearing is located at the end of the shaft, stresses near the bearing are often not critical since the bending moment is small.

1. Axle

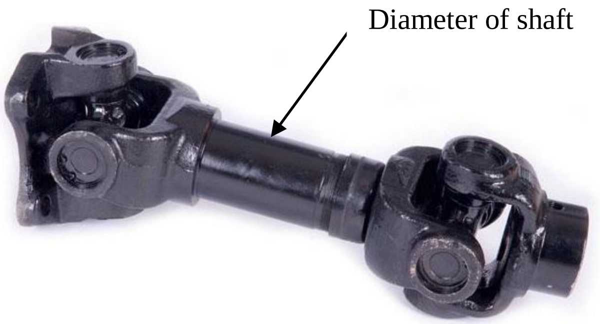
The axle is a non-rotating machine part, which is not loaded with any torque and serves to support rotating wheels, pulleys and the like. There is also a wide use of axles in the field of motoring, train transport and other types of transport.



The non-rotating axle can be easily designed and analyzed as a static circular beam.

2. Connecting shafts

Connecting shafts are also considered to be types of shafts which serve to connect different types of devices. The connecting shaft transmits the rotational movement from one device to another and is loaded only with torque. A practical example of a connecting shaft is shown in fig.



3. Universal shafts

Shafts referred to as universal are widely used in various types of equipment. These are shafts that are stressed by a combination of torque and bending moment. Such shafts are input, countershaft, output shafts of gearboxes, various shafts of driven axles and other mechanisms, etc.

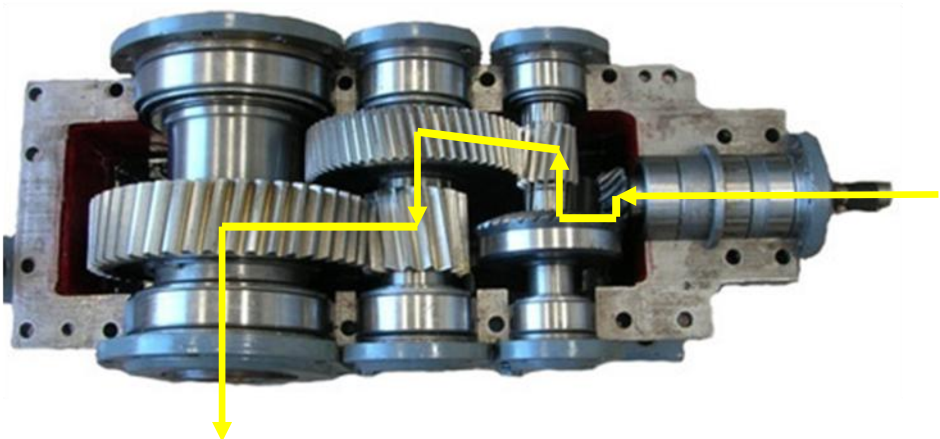


Fig. Power flow in a multi-speed multi-speed transmission

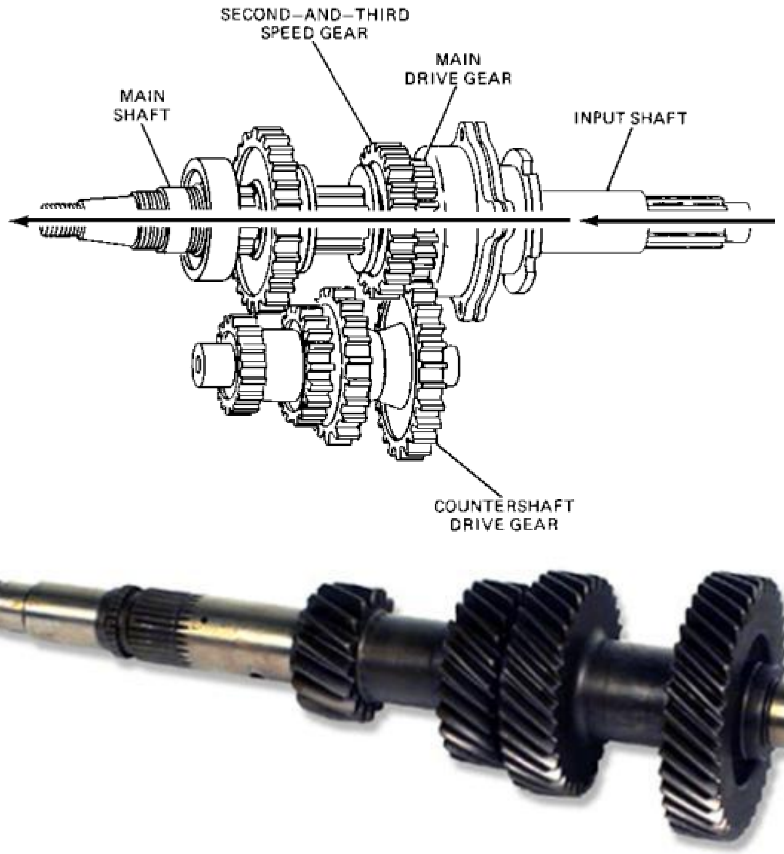


Fig. Multi-speed transmission shaft

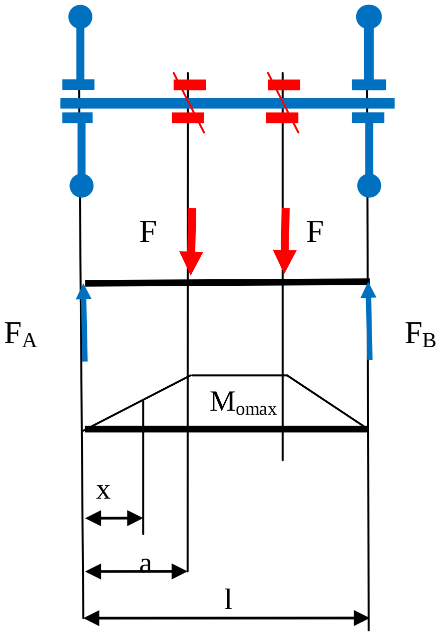
When designing these shafts, it is necessary to focus primarily on critical points. Critical points will usually be on the outer surface of the shafts, at points of axial load, at places where the bending moment is large and is combined with torque, and where stress concentrations exist (keyways, transitions, recesses, threads, etc.). By direct comparison of various such places along the length of the shaft, it is possible to identify several critical places where the design and inspection of the shaft should focus on:

a.) Stress checking
b.) Deformation checking
c.) Checking of vibration and critical shaft speed.
d.) Fatigue strength control

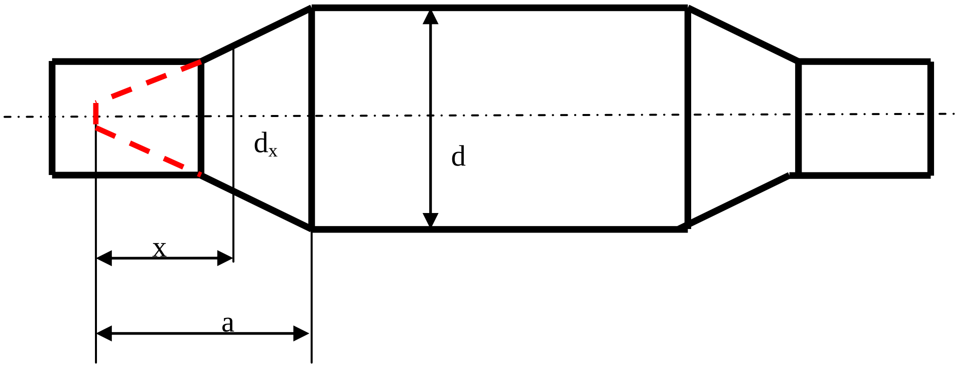
The critical points for the stress, fatigue and deformation control of the shaft do not have to be identical.

Axle – nonrotating shaft, load only bending moment

Calculation model for axle:



Design of shaft with using Stress Optimization



Connecting shaft – only rotating shaft, load only torque

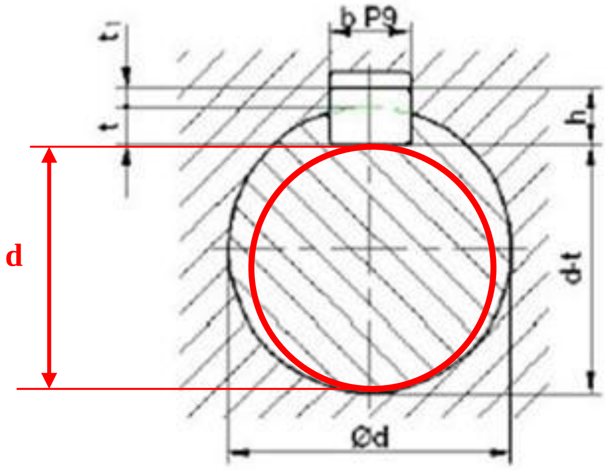
d_0 - internal diameter of shaft

Diameter for full cross section shaft:

$$d = \sqrt[3]{\frac{M_k}{0{,}2.\tau_D}} = \sqrt[3]{\frac{5.M_k}{\tau_D}}$$

τ_D – usually 12-30MPa

As a result of using a key or pins or other shaped element to connect the shaft to the hub, the actual diameter of the shaft is reduced. When designing, it is then necessary to consider a weakened shaft cross-section and the design of the diameter will be a design of a weakened shaft cross-section. (red circle)



Stiffness condition:

$$\varphi = \frac{M_k.l}{G.J_p} \leq \varphi_D$$

φ - twist angle [rad]
l – length of shaft [m]

Limit twist angle:

$$\varphi_D{}^{\cap} = \frac{\pi}{180}.\varphi_D{}^{\circ}$$

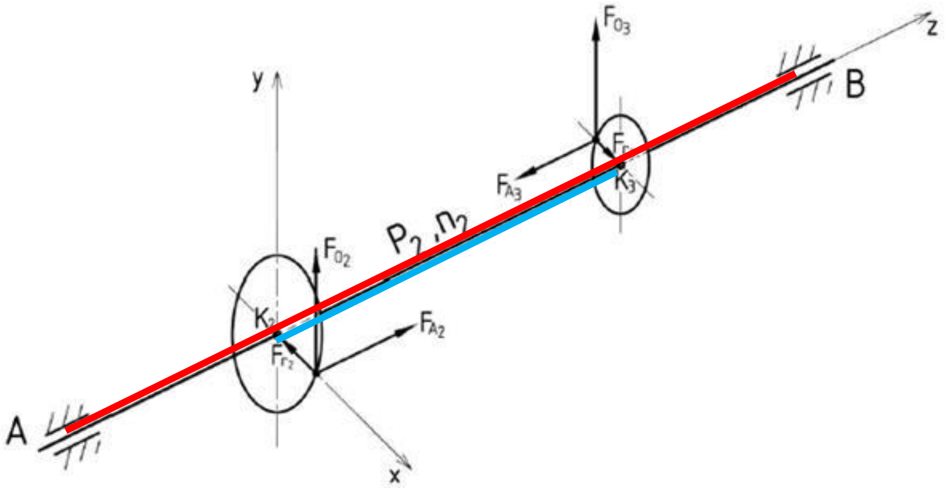
φ_D° - limit twist angle [°] – usually φ_D° = 0,25°/m

Universal shaft – combinations load with bending moments and torque

In gear box are shafts load with combination bending moment and torque usually.

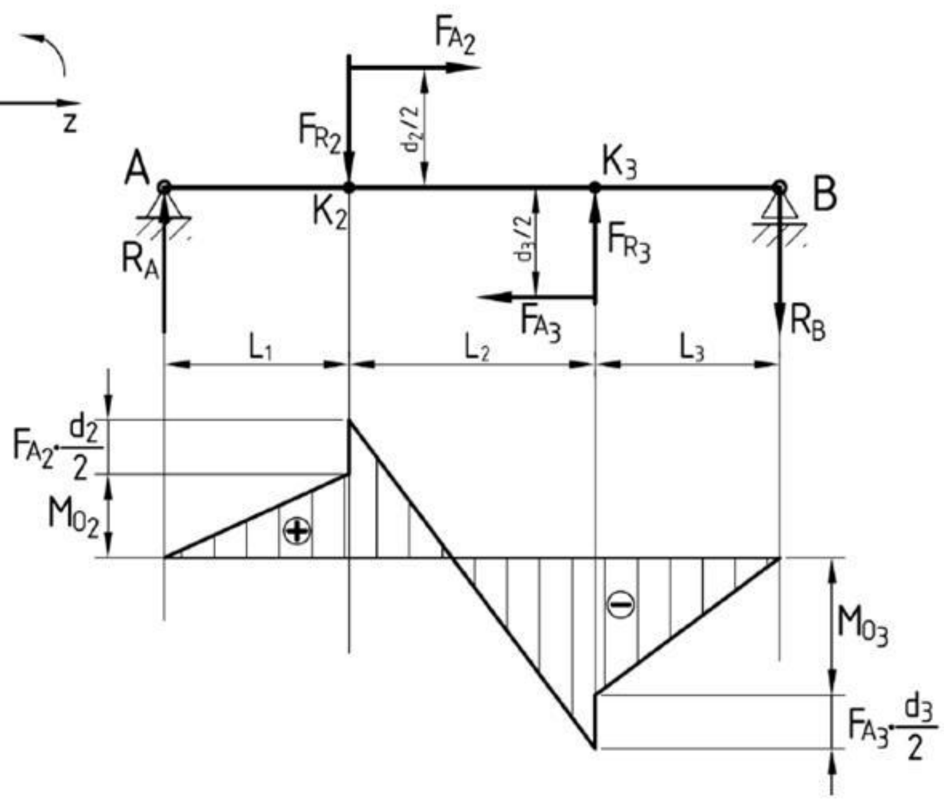
a.) Shaft stress checking

The scheme of the countershaft and the method of its loading



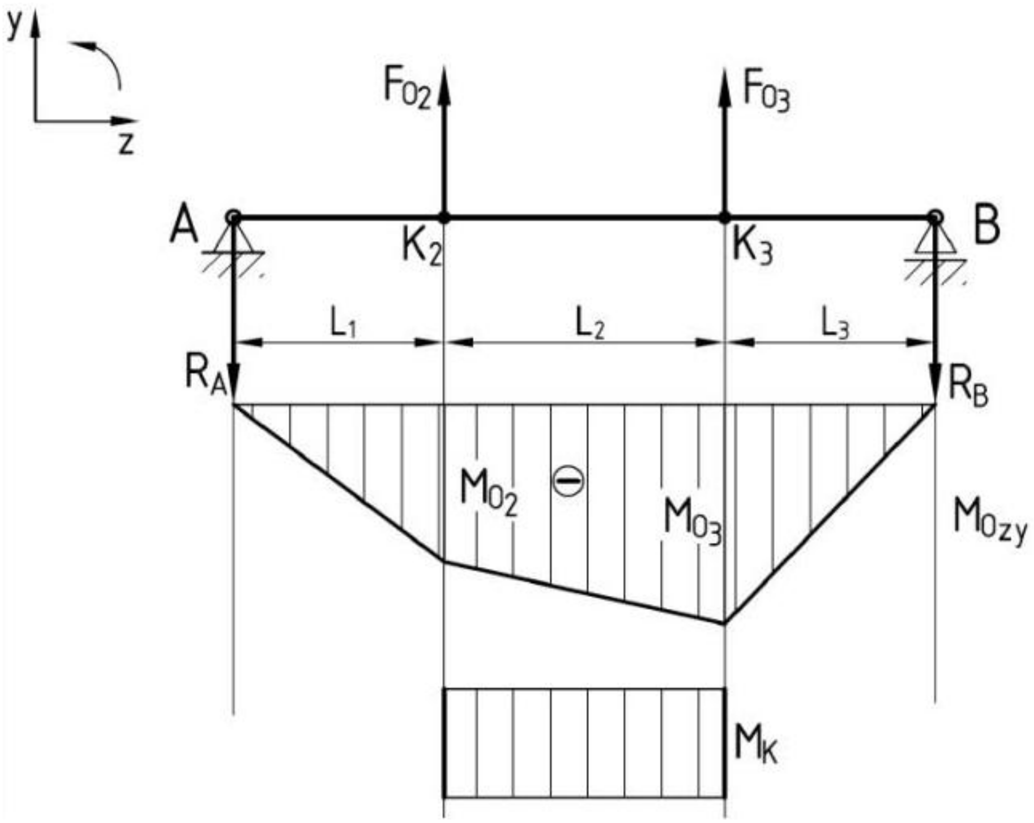
When:
F_o – tangential force
F_r – radial force
F_A – axial force

The space load of the shaft must be solved by superposition in the planes, and look for a solution in two planes.
<u>Numerical model in plane z-x</u>



Axial forces acting on wheel radius create constant bending moments that can be defined:

Numerical model in plane z-y



Calculation of reaction in points A and B:

The course of the bending moment is determined from known conditions using the cutting method. The points with the maximum value of the bending moment are the points K2 and K3 and the bending moment at these points is determined as follows:

The resulting bending moment at points K2 and K3 is determined as follows:

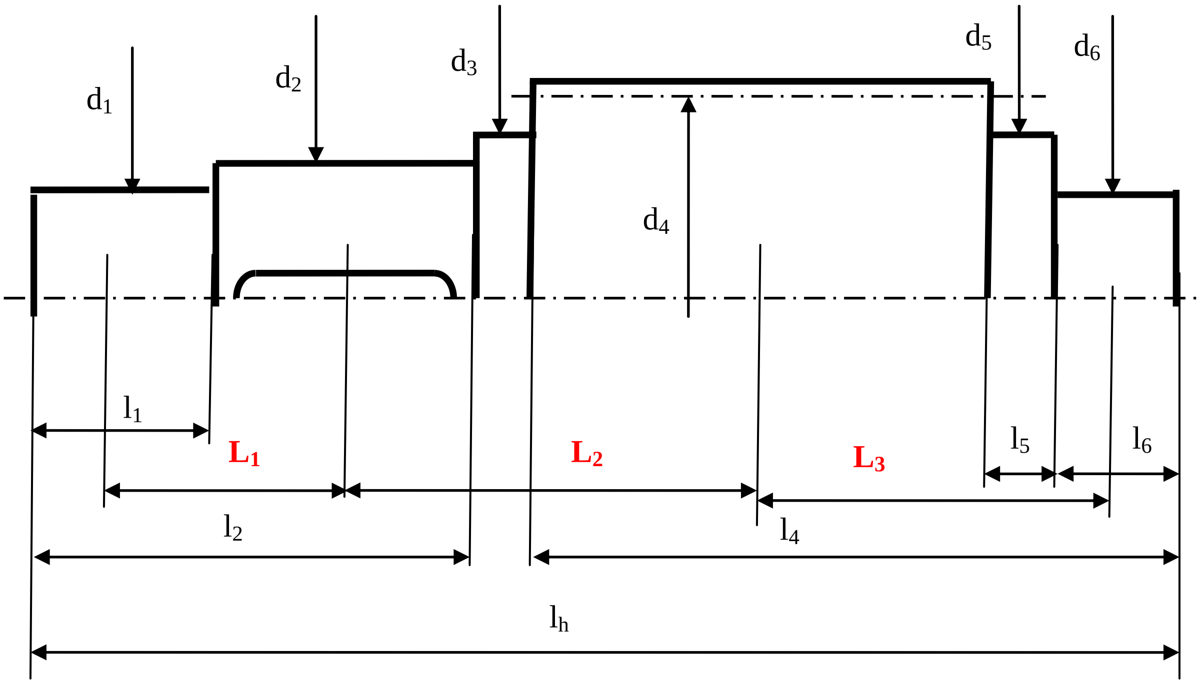
Between points K2 and K3 there is a section of the shaft stressed by torque. As we can see from the course, the torque value is constant.

Strain conditions:

Bending stress:

Finally stress for generally position of shaft:

Finally design of shaft



Recommendation for determining the dimensions and shape of the shaft:
1. From the stress conditions we calculate the diameter of the shaft at critical points (K2 and K3)
2. We move from critical places to places of reactions, ie to places where there will be deposits.
3. Reducing the dimensions of the shaft on the other sections must be realized gradually in terms of the resulting course of bending stress.
4. The places where the bearings will be must have diameter values identical to the standardized (normalized) values of the diameters of the inner rings of the bearings.

b.) Shaft deformation checking

Deflection Considerations

Deflection analysis at even a single point of interest requires complete geometry information for the entire shaft. For this reason, it is desirable to design the dimensions at critical locations to handle the stresses, and fill in reasonable estimates for all other dimensions, before performing a deflection analysis. Deflection of the shaft, both linear and angular, should be checked at gears and bearings. Allowable deflections will depend on many factors, and bearing and gear catalogs should be used for guidance on allowable misalignment for specific bearings and gears. As a rough guideline, typical ranges for maximum slopes and transverse deflections of the shaft centerline are given in Table 8.1.

Table 8.1

Typical Maximum Ranges for Slopes and Transverse Deflections

Slopes	
Tapered roller	0.0005–0.0012 rad
Cylindrical roller	0.0008–0.0012 rad
Deep-groove ball	0.001–0.003 rad
Spherical ball	0.026–0.052 rad
Self-align ball	0.026–0.052 rad
Uncrowned spur gear	<0.0005 rad
Transverse Deflections	
Spur gears with $P < 10$ teeth/in	0.010 in
Spur gears with $11 < P < 19$	0.005 in
Spur gears with $20 < P < 50$	0.003 in

Check shaft deflection and slopes

As already mentioned, the solution and design of the shaft is a space task due to the applied load. In the case of a space problem, the deflection line of the shaft is also space. The resulting deflection and the resulting slopes angles can be determined by the vector sum of the partial solved planes perpendicular to each other as follows:

Finally deflections:

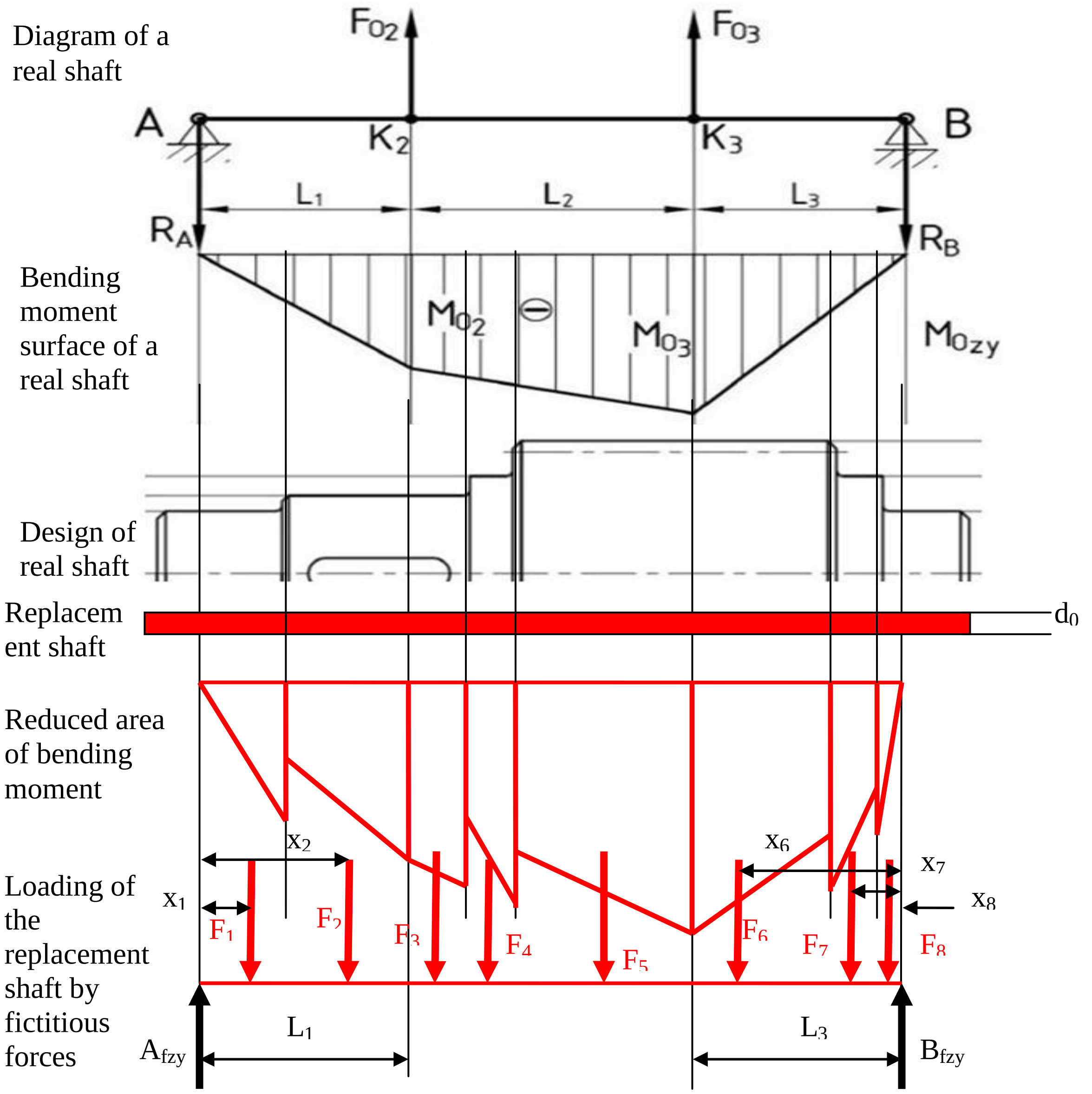
$$y_i = \sqrt{{y_{izy}}^2 + {y_{izx}}^2}$$

Finally slopes:

$$\alpha_i = \sqrt{{\alpha_{izy}}^2 + {\alpha_{izx}}^2}$$

Numerical solution of the deflection line in two planes is quite demanding and therefore the use of graphical - analytical Maxwell - Mohr method was used in practice. The principle of this method is based on the principle of replacing the actual shape of a shaft consisting of different diameters, by means of a replacement rod of one diameter. This replacement is carried out provided that the deformation is maintained, where the deflection of the replacement rod is identical to the deflection of the actual shaft. If this condition is to be met, the torque area of the actual shaft which causes this deformation must be reduced and replaced by a reduced spare torque surface stressing the spare rod of constant diameter.
The procedure can be summarized in the following steps:

1. Determination of second-area moments on all diameters of a real shaft.
2. Determination of second-area moments of a replacement shaft - spare rod.
3. Determination of values of reduced bending moment.
4. Plotting the course of the reduced bending moment on the spare rod.
5. Determination of fictitious forces causing the course of the reduced bending moment.
6. Determining the scope of fictitious forces.
7. Determination of fictitious reactions on a replacement rod.
8. Determination of fictitious bending moment, by the method of cutting at a critical point.
9. Determination of deflection from the deflection equation.
10. Determination of the slopes angle at the reaction site.



1. Determination of second-area moments on all diameters of a real shaft.

$$I_i = \frac{\pi}{64}\, d_i^{\,4}$$

2. Determination of second-area moments of a replacement shaft - spare rod.

$$I_0 = \frac{\pi}{64}\, d_0^{\,4}$$

From ratio of second-area moments:

$$\frac{I_0}{I_i} = \left(\frac{d_0}{d_i}\right)^4$$

3. Determination of values of reduced bending moment.

$$M_{ozy}{}' = M_{ozy} . \frac{I_0}{I_i}$$

5. Determination of fictitious forces causing the course of the reduced bending moment.

The fictitious force replacing the reduced moment surface still acts at the center of gravity of the replaced surface and is determined as follows:

If surface is triangle:

$$F_i = \frac{1}{2} M_{oizy}{}' . \frac{I_0}{I_i} . l_i$$

If surface is rectangle:

$$F_i = \frac{1}{2} (M_{oizyL}{}' - M_{oizyP}{}') \ . \frac{I_0}{I_i} . l_i$$

When:

l_i – the length of the section of the replacement reduced moment surface

6. Determining the scope of fictitious forces.

The point of application of the fictitious force is at the center of gravity of the area from which it was determined and is determined by dimensional analysis.

7. Determination of fictitious reactions on a replacement rod..

When determining fictitious reactions, we start from the moment condition of equilibrium to point A and to point B.

$$\sum M_{fiA} = 0$$

$$\sum M_{fiB} = 0$$

From equilibrium conditions

$$B_{fzy} = \frac{F_1 . x_1 + F_2 . x_2 + F_3 . x_3 + \cdots + F_n . x_n}{L}$$

$$A_{fzy} = \frac{F_1 . (L - x_1) + F_2 . (L - x_2) + \cdots + F_n . (L - x_n)}{L}$$

Also:

$$A_{fzy} + B_{fzy} = \sum F_i$$

8. Determination of fictitious bending moment, by the method of cutting at a critical point..

$$M_{fizy} = \sum F_i \,.\, z_i$$

Fictitious bending moment in point K_3:

$$M_{f3zy} = B_{fzy}.L_3 - F_8.(L_3 - x_8) - F_7.(L_3 - x_7) - F_6.(L_3 - x_6)$$

Fictitious bending moment in point K_2:

$$M_{f2zy} = A_{fzy}.L_1 - F_1.x_1 - F_2.x_2$$

9. Determination of deflection from the deflection equation.

$$y_{izy} = \frac{M_{fizy}}{E.I_0}$$

Deflection in point K_2:

$$y_{2zy} = \frac{M_{f2zy}}{E.I_0}$$

Deflection in point K_3:

$$y_{3zy} = \frac{M_{f3zy}}{E.I_0}$$

10. Determination of the slopes angle at the reaction site.
The slope angle is determined at the reaction site, i.e. at the bearing location of the shaft in the bearings.
Finally fictitious reactions:

$$A_f = \sqrt{{A_{fzy}}^2 + {A_{fzx}}^2}$$

$$B_f = \sqrt{{B_{fzy}}^2 + {B_{fzx}}^2}$$

Finally slopes:

$$\alpha_A = \frac{A_f}{E.I_0}$$

$$\alpha_B = \frac{B_f}{E.I_0}$$

Checking of twist angle

It only makes sense to check the torsion of the shaft on the part of the shaft that is actually twisted. This means that we only check the section of the shaft between the input and output of torque from the shaft.

Stiffness condition:

$$\varphi = \frac{M_k.l}{G.J_p} \leq \varphi_D$$

φ - twist angle [rad]
l – length of shaft [m]

Limit twist angle:

$$\varphi_D{}^{\cap} = \frac{\pi}{180} . \varphi_D{}^{\circ}$$

$\varphi_D{}^{\circ}$ - limit twist angle [°] – usually $\varphi_D{}^{\circ}$ = 0,25°/m
$\varphi_D{}^{\circ}$ - $0{,}25.10^{-2}$ rad/m – shock load
$\varphi_D{}^{\circ}$ - $0{,}5.10^{-2}$ rad/m – fluctuating load
$\varphi_D{}^{\circ}$ - $(1\text{-}2).10^{-2}$ rad/m – static load

c.) Checking of vibration and critical shaft speed

As we have seen above topic b, real shafts are characterized by a certain rigidity and, in connection with the material disks located on the shaft (gears, pulleys, flanges, etc.), are able to oscillate. Depending on the mode of oscillation, it is possible to monitor the oscillation of the shafts:

1. Deflection oscillation
2. Circular oscillation
3. Torsional oscillation

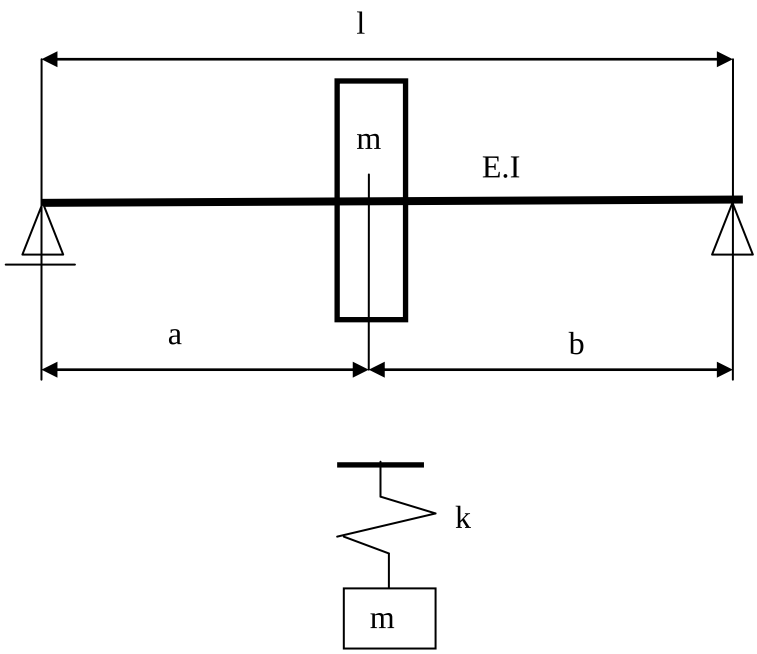
Due to a one-time pulse, the shaft oscillates at its own frequency, which we denote Ω. However, if periodic excitation pulses whose frequency is act on the shaft, then these pulses are called the excitation frequency. If the value of the excitation frequency is equal to the value of the natural frequency ω=Ω a state called resonance occurs. The state of resonance is a very dangerous state because it causes a resonant amplification of the excitation which is manifested by an excessive load and many times this excessive load leads to a shaft failure. For this reason, it is recommended that the shafts operate at least 20% away from resonance.

Then, if the ratio of the excitation frequency to the natural frequency is defined by the tuning factor η, then it must apply to each of the modes of oscillation: 1.2 <η <0.8.

1. Deflection oscillation

When a shaft is turning, eccentricity causes a centrifugal force deflection, which is resisted by the shaft's flexural rigidity EI. As long as deflections are small, no harm is done. Another potential problem, however, is called critical speeds: at certain speeds the shaft is unstable, with deflections increasing without upper bound. It is fortunate that although the dynamic deflection shape is unknown, using a static deflection curve gives an excellent estimate of the lowest critical speed. Such a curve meets the boundary condition of the differential equation (zero moment and deflection at both bearings) and the shaft energy is not particularly sensitive to the exact shape of the deflection curve. Designers seek first critical speeds at least twice the operating speed. The shaft, because of its own mass, has a critical speed. The ensemble of attachments to a shaft likewise has a critical speed that is much lower than the shaft's intrinsic critical speed. Estimating these critical speeds (and harmonics) is a task of the designer.

During deflection oscillation, the shaft oscillates in the plane of the deflection line. To calculate the natural circular frequency of an intangible shaft with a disk of mass *m*, it is possible to make a replacement oscillating system so that the stiffness of the shaft will represent the stiffness of the spring *k*, on which the mass of mass of the disk *m* is suspended.



The stiffness of the substitute spring can be expressed:

$$k = \frac{3.E.I.l}{a^2.b^2}$$

Then, if the damping is neglected, we can write the equation of motion in the form:

$$m.\ddot{y} + k.y = 0$$

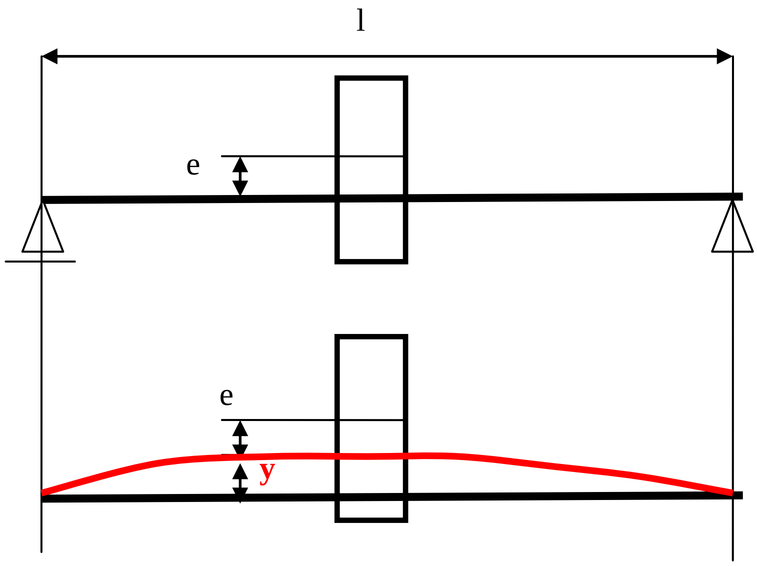
By solving the equation of motion, we get the relation for the natural frequency of the shaft:

$$\Omega = \sqrt{\frac{k}{m}}$$

From the natural frequency it is possible to express the frequency parameter $\lambda = \Omega^2$. The values of the frequency parameter can be found in the recommended literature and depend on the way the shaft is mounted and on the shape of the deflection line - ie on the shape of the oscillation.

2. Circular oscillation

In the case of circular oscillations, the shaft is deformed due to the centrifugal force caused by the shaft imbalance, FIG.



From the condition of force balance between external centrifugal force and internal deformation force:

$$m.\omega^2.(e + y) = k.y$$

The deflection of the shaft can be expressed:

$$y = \frac{m.e.\omega^2}{k - m.\omega^2}$$

We get the relation for the natural frequency of the shaft by the deflection equation:

$$\omega_{kr} = \Omega = \sqrt{\frac{k}{m}}$$

If we express the dependence of the ratios y/e on ω / ω_{kr}, we can write a relation for the approximate value of the critical shaft speed resp. critical shaft frequency as follows:

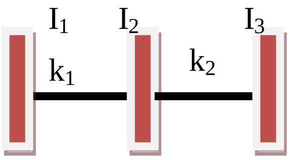
$$\omega_{kr} = \frac{15{,}8}{\sqrt{y_{stat}}}$$

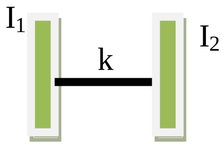
$$n_{kr} = \frac{946}{\sqrt{y_{stat}}}$$

if:
y_{stat} – static deflection at the disk location (mm)

3. Torsional oscillation

Any rotating system of mass disks which are interconnected can be considered as a torsionally oscillating system. The excitation pulse which is introduced into such a system depends on the excitation frequency which may have the character of a harmonic component. Each n-mass torsion system can be reduced to three-mass or the two-mass torsionally oscillating system of FIG.





Then the natural frequency for a two-mass system without damping can be expressed:

$$\Omega = \sqrt{\frac{k}{I_{red}}}$$

Where:
k - torsional stiffness
I_{red} - reduced mass moment of inertia of the system

$$I_{red} = \frac{I_1 . I_2}{I_1 + I_2}$$

The additional dynamic moment arising from the influence of torsional oscillations can be expressed:

$$M_d = \frac{M_i . I_2}{I_1 + I_2} . \frac{1}{\left[1 - \left(\frac{i.\omega}{\Omega}\right)\right]}$$

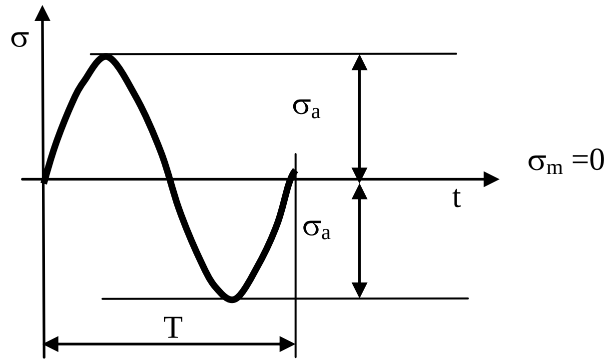
Where:
M_i - moment amplitude of the harmonic component of the i-th order

Resonance occurs if: i.ω = Ω - And the maximum shaft torque will be given by the sum of the static component and the dynamic component as follows:

$$M_{max} = M + M_d$$

d.) Fatigue strength control

If the external load causing the bending stress of the shaft is constant, then due to the rotation of the shaft, it will be stressed by fluctuating bending stress. The fluctuating symmetrical course from bending stress



Maximum stress:

$$\sigma_h = \sigma_m + \sigma_a$$

Minimum stress:

$$\sigma_n = \sigma_m - \sigma_a$$

Midrange component of stress:

$$\sigma_m = \frac{\sigma_h + \sigma_n}{2} = \frac{\sigma_{max} + \sigma_{min}}{2}$$

Amplitude component:

$$\sigma_a = \frac{\sigma_h - \sigma_n}{2} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

Stress ratio:

$$r = \frac{\sigma_n}{\sigma_h}$$

The limit state of the shaft stress is the state when the mean value of the stress will be constant or zero and will increase only by the amplitude of the stress. The value of the midrange stress is influenced by the unbalance of the shaft. The amplitude stress is influenced by the external load of the shaft. To determine the degree of safety, we use the known procedure, see Smith's diagrams.

In the case of shaft stress, it is a common case that the midrange stress also increases with increasing amplitude stress so that their ratio σ_a / σ_m does not change. We will use the known procedure to determine the degree of safety, see. Smith diagrams.

Where: After taking into account the above factors, the fatigue limit for the real component will be:

Fluctuating bending:

$$\sigma_c{}^* = \frac{\sigma_c . \nu_\sigma . \varepsilon_p}{\beta_\sigma}$$

Fluctuating torque:

$$\tau_c{}^* = \frac{\tau_c . \nu_\tau . \varepsilon_p}{\beta_\tau}$$

Safety parameter of bending:

$$k_o = \frac{\sigma_c}{\sigma_a}$$

Safety parameter of torque:

$$k_k = \frac{\tau_c}{\tau_a}$$

Finally safety parameter:

$$k = \frac{k_o . k_k}{\sqrt{(k_o^2 + k_k^2)}}$$

Recommended minimum values for finally safety parameter:
k = 1,3 - 1,5 - at very precisely determined stress, specific production conditions and precisely observed conditions of use
k = 1.5 - 1.8 - at less precise stress, less precise production conditions and less precise conditions of use
k = 1.8 - 2.5 - gross accuracy of the specified stress, rough production conditions and for shafts with diameters more than 200 mm.